

Event-Based Estimation

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1 Problem Formulation

Let $x \in \mathbb{R}^n$ be the state, such that $x \sim \mathcal{N}(\bar{x}, \Sigma)$. $y \in \mathbb{R}^m$ is the sensor measurement, where y_i is the measurement from the i th sensor, such that

$$y_i = a_i x + v_i.$$

Assume that x, v_1, \dots, v_m are all linearly independent and $v_i \sim \mathcal{N}(0, r_i)$.

Let γ_i be a binary variable, such that $\gamma_i = 0$ if the sensor is not selected. $\gamma_i = 1$ if the sensor is selected.

In the previous lecture, we consider γ_i to be independent from y_i .

In this lecture, we will assume γ_i depends on y_i .

Consider the sensors employ the following strategy:

1. Each sensor randomly generates a uniformly distributed random variable ξ_i on $[0, 1]$.
2. Sensor i computes a function:

$$g_i(y_i) = \exp\left(-\frac{y_i^2}{2\delta_i}\right),$$

where $\delta_i > 0$ is a fixed parameter.

3. Sensor i decides whether to send the measurement or not:

$$\gamma_i = \begin{cases} 0 & \text{if } \xi_i \leq g_i(y_i) \\ 1 & \text{if } \xi_i > g_i(y_i) \end{cases}$$

Define the information set $\mathcal{I} = \{\gamma_1, \dots, \gamma_m, \gamma_1 y_1, \dots, \gamma_m y_m\}$ and state estimation and the estimation error covariance matrix as $\hat{x} \triangleq \mathbb{E}(x|\mathcal{I})$, $P \triangleq \text{Cov}(x|\mathcal{I})$ respectively.

2 Optimal Estimator

Consider single sensor case. The joint probability of x, y, γ satisfies

$$\begin{aligned} P(x \in X, y \in Y, \gamma = 0) &= P(x \in X)P(y \in Y|x \in X)P(\gamma = 0|y \in Y) \\ &= \int_{x \in X} \int_{y \in Y} \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp\left(-\frac{1}{2}(x - \bar{x})^T \Sigma^{-1}(x - \bar{x})\right) \frac{1}{\sqrt{2\pi r}} \exp\left(-\frac{1}{2r}(y - ax)^2\right) g(y) dx dy. \end{aligned}$$

Hence,

$$\begin{aligned} P(x \in X, \gamma = 0) &= \alpha \int_{x \in X} \int_{y \in \mathbb{R}} \exp\left[-\frac{1}{2}\left((x - \bar{x})^T \Sigma^{-1}(x - \bar{x}) + \frac{1}{r}(y - ax)^2 + \frac{1}{\delta}y^2\right)\right] dx dy \\ &= \alpha \int_{x \in X} \exp\left[-\frac{1}{2}(x - \bar{x})^T \Sigma^{-1}(x - \bar{x})\right] \int_{y \in \mathbb{R}} \exp\left[-\frac{1}{2}\left(\frac{1}{r}(y - ax)^2 + \frac{1}{\delta}y^2\right)\right] dy dx \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{r}(y - ax)^2 + \frac{1}{\delta}y^2 &= \left(\frac{1}{r} + \frac{1}{\delta}\right)y^2 - \frac{2}{r}(ax)y + \frac{1}{r}(ax)^2 \\ &= \left[\sqrt{\frac{r + \delta}{r\delta}}y - \sqrt{\frac{\delta}{r(r + \delta)}}(ax)\right]^2 + \frac{1}{r + \delta}x^T a^T a x. \end{aligned}$$

Hence,

$$\begin{aligned} P(x \in X, \gamma = 0) &= \beta \int_{x \in X} \exp\left[-\frac{1}{2}\left((x - \bar{x})^T \Sigma^{-1}(x - \bar{x}) + \frac{1}{r + \delta}x^T a^T a x\right)\right] dx \\ &= \lambda \int_{x \in X} \exp\left[-\frac{1}{2}(x - \tilde{x})^T \tilde{P}^{-1}(x - \tilde{x})\right], \end{aligned}$$

where

$$\tilde{P} = \left(\Sigma^{-1} + \frac{a^T a}{r + \delta}\right)^{-1} = \Sigma - \Sigma a^T (a \Sigma a^T + r + \delta)^{-1} a \Sigma.$$

and

$$\tilde{x} = P \Sigma^{-1} \bar{x} = \bar{x} + \Sigma a^T (a \Sigma a^T + r + \delta)^{-1} (0 - a \bar{x}).$$

Hence, if $P(x|\gamma = 0)$ is Gaussian distributed with mean \tilde{x} and covariance P .

On the other hand,

$$\begin{aligned} P(x \in X|y \in Y, \gamma = 1) &= \frac{P(\gamma = 1|x \in X, y \in Y)P(x \in X|y \in Y)}{P(\gamma = 1|y \in Y)} = \frac{(1 - g(y))P(x \in X|y \in Y)}{1 - g(y)} \\ &= P(x \in X|y \in Y). \end{aligned}$$

Hence, $P(x|y, \gamma = 1)$ is a Gaussian distribution with mean $\bar{x} + \Sigma a^T (a \Sigma a^T + r)^{-1}(y - a\bar{x})$ and variance $P = (\Sigma^{-1} + a^T a/r)^{-1}$.

Therefore

$$\hat{x} = \bar{x} + \Sigma a^T (a \Sigma a^T + r + (1 - \gamma)\delta)^{-1}(\gamma y - a\bar{x}),$$

and

$$P = \left(\Sigma^{-1} + \frac{a^T a}{r + (1 - \gamma)\delta} \right)^{-1}$$

For multi-sensors, define

$$\Gamma = \text{diag}(\gamma_1, \dots, \gamma_m), \Delta = \text{diag}(\delta_1, \dots, \delta_m).$$

then

$$\hat{x} = \bar{x} + \Sigma C^T (C \Sigma C^T + R + (I - \Gamma)\Delta)^{-1} (\Gamma y - C \bar{x}),$$

and

$$P = \left(\Sigma^{-1} + \sum_{i=1}^m \frac{a_i^T a_i}{r_i + (1 - \gamma_i)\delta_i} \right)^{-1} = \Sigma - \Sigma C^T (C \Sigma C^T + R + (I - \Gamma)\Delta)^{-1} C \Sigma.$$

3 Communication Rate

Consider single sensor case and assume that $\bar{x} = 0$. Let us define the communication rate $\lambda = P(\gamma = 1)$. Then

$$P(\gamma = 0) = \mathbb{E} P(\xi \leq g(y)|y) = \mathbb{E} g(y).$$

y is Gaussian with mean 0 and variance $r + a \Sigma a^T$. Hence,

$$\begin{aligned} \mathbb{E} g(y) &= \frac{1}{\sqrt{2\pi(r + a \Sigma a^T)}} \int_{y \in \mathbb{R}} \exp \left[-\frac{1}{2} \left(\frac{y^2}{r + a \Sigma a^T} + \frac{y^2}{\delta} \right) \right] \\ &= \frac{1}{\sqrt{1 + \frac{r + a \Sigma a^T}{\delta}}}. \end{aligned}$$

For multi sensor case,

$$\lambda_i = 1 - \frac{1}{\sqrt{1 + \frac{r_i + a_i \Sigma a_i^T}{\delta_i}}}.$$