Complexity and Fragility in Linear Programming

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Outline

Motivation

- In analysis and design of complex systems, we often need to prove robustness, e.g.,
  - formal verification of programs and protocols
  - robustness analysis of dynamics of biological networks

- such questions are typically computationally **intractable**
- conventional methods provide little encouragement that this can be systematically overcome
- not much hope *if* **robust** instances are as hard as worst-case…

Motivation

how does robustness affect complexity?

conjecture:

(proof) complexity implies fragility

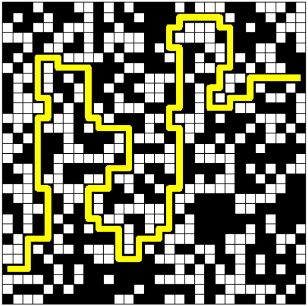
this talk:

three case studies support the conjecture: Linear Programming (P), Mandelbrot set membership (undecidable), number partitioning (NP hard)

From lattice to LP

Monday's talk: 'complexity implies fragility' in lattice percolation (which can be cast as LP)

- **complexity** = length of shortest path/barrier
- **fragility** = smallest change in sites' color that changes connectivity



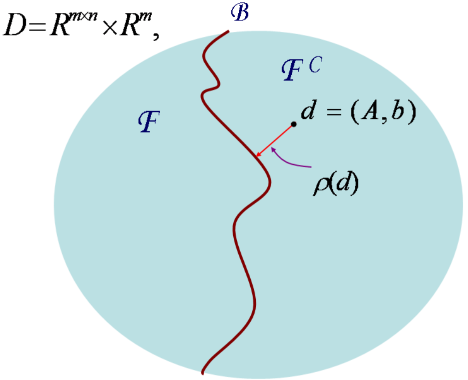
are there natural notions of complexity and fragility for general LPs? does the conjecture extend?

Outline

Fragility in LP

LP feasibility problem: $\exists\, x \in \mathbb{R}^n$ s.t. $Ax \preceq b, \quad x \succeq 0$?
(A, b) are instance data.

- instances on boundary $\mathcal{B}$: **ill-posed** (arbitrarily small change in data can change feasibility status)
- instances near boundary: **ill-conditioned**



Condition number as fragility

distance to ill-posedness

for every instance (A, b),

$$\rho(d) = \mathbf{dist}(d, \mathcal{B}) = \inf\{\|d - \tilde{d}\| : \tilde{d} \in \mathcal{B}\},$$

where $\|d\| = \max\{\|A\|, \|b\|\}$.

- **robustness:** normalized (scale-invariant) distance to ill-posedness: $R = \rho(d)/\|d\|$
- **fragility:** $F = 1/R$, known as Renegar's condition number.

Condition number

a ubiquitous notion in numerical analysis

simplest example:
for $A \in \mathbb{R}^{n \times n}$, $\kappa(A) = \lambda_{\max}/\lambda_{\min}$ is inverse of normalized distance to singularity. many applications:

- bounds sensitivity of A^{-1} w.r.t. perturbation in A
- bounds change in solution size x of $Ax = b$ w.r.t perturbation in b and A (separately)
- bounds computational cost of many numerical operations

Renegar's LP condition number has similar properties

Complexity (proof length) in LP

for complexity of an instance, we can take the smallest "proof size":

- proof of feasibility: present a feasible x; can take proof size = $\|x\|$,

$$C = \inf\{\|x\| \mid Ax \preceq b,\ x \succeq 0\}$$

- proof of infeasibility: present a dual feasible λ; can take proof size = $\|\lambda\|$

$$C = \inf\{\|\lambda\| \mid \underbrace{A^T\lambda \succeq 0}_{\succeq \alpha},\ \underbrace{b^T\lambda < 0}_{\leq -\beta},\ \lambda \succeq 0\}$$

(strong duality holds for LP, excluding (pathological) case where both primal and dual are infeasible. $\alpha > 0$, $\beta > 0$ allow us to deal with the boundary separately)

Outline

Complexity implies fragility

Theorem [Renegar '94]

if d feasible and not on boundary, there exists feasible x_0 s.t.

$$\|x_0\| \leq \frac{\|d\|}{\rho(d)}.$$

if d infeasible, can show the result extends to the *alternative* LP, i.e., there exists λ_0 s.t.

$$\|\lambda_0\| \leq \frac{\|d\|}{\rho(d)}.$$

thus, $C \leq \|x_0\| \leq F$, or $C \leq \|\lambda_0\| \leq F$.

Condition number F and algorithm complexity

- for an ellipsoid method in [Freund,Vera '98] (with objective fct $c^T x$, ϵ-optimal)

$$\#\text{of iterations} \leq \mathcal{O}\left(n^2 \log\left(\frac{F\|c\|_*}{c_1 \epsilon}\right)\right)$$

(c_1 on order of number of constraints m)

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- for an interior point method (barrier) in [Renegar '94] (feasibility problem):

$$\#\text{of iterations} \leq \mathcal{O}\left(K \log(K + F + \ldots)\right)$$

  K is the barrier parameter; other terms depend on initial point

complexity essentially bounded by logarithmic function of F

Outline

1. Motivation
2. Fragility and condition number
3. Complexity implies fragility
4. **Summary**

More properties of condition number

- for any LP, a slight random relative perturbation has small condition number with *high probability* ([Dunagan,Spielman,Teng '02])
- many results extend to more general *convex* problems in **conic linear form**, that includes semidefinite programming...

Summary:

- with condition number as 'fragility', we have $C \leq F$ for LPs
- condition number also bounds iteration number for various algorithms