Scale-Independent Proofs in Systems and Control

(title by John)

THE STRAIGHTJACKET OF DIMENSION FREE STRUCTURES

– SHORT FORMULAS –

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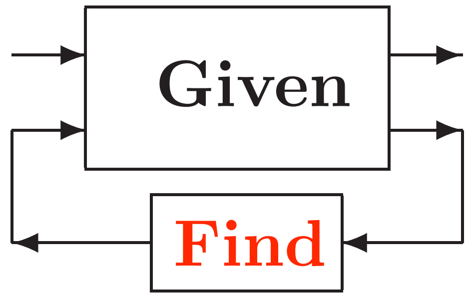
NCAlgebra[a]

[a] Helton, Stankus (CalPoly SanLObispo), Miller (Archemedies Inc)

Systems problems $\longrightarrow$ Matrix Ineq



Many such problems Eg. H^∞ control

Example: Get Riccati expressions like

$$AX + XA^T - XBB^TX + CC^T \succ 0$$

OR Linear Matrix Inequalities (LMI) like

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \succ 0$$

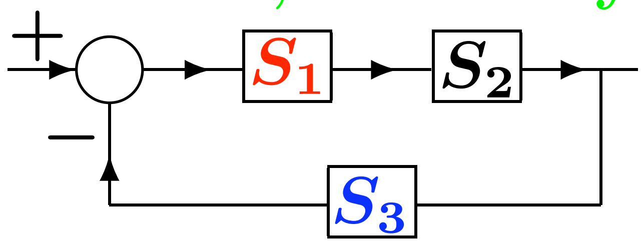
which is equivalent to the Riccati inequality.

ncDimlessPart

DIMENSIONLESS FORMULAS

1. **DIMENSIONLESS FORMULAS – THIS TALK**

   Topology is fixed; but many systems . E.g.



WANT FORMULAS: which hold regardless of the dimension of system S_1, S_2, S_3. Then unknowns are matrices and formulas respect matrix multiplication.

Eg. Most classical control text problems:

Control pre 1990: Zhou,Doyle,Glover.

LMIs in Control: Skelton, Iwasaki,

. Grigoriadis.

– Get noncommutative formulas

Matrices Whole

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \succeq \mathbf{0} \tag{1}$$

Looks the same regardless of system size.

Keeping Matrices Whole

Matrices Whole

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \succeq \mathbf{0} \qquad (1)$$

Looks the same regardless of system size.

Matrices Entry by Entry – "Disaggregated"

If dimensions of the matrices A, B, C, X are specified, we can write formula (1) with matrices $L_0, \ldots, L_m$ as

$$\sum_{j=0}^{m} L_j \, X_j \succeq 0$$

with the unknown numbers X_j taken as entries of X.

Disaggregated Matrices

Example: If $A \in \mathbb{R}^{2\times 2}, B \in \mathbb{R}^{2\times 1}, C \in \mathbb{R}^{1\times 2}$, then $X \in \mathbb{S}^2$ and we would take

$$X = \begin{pmatrix} X_1 & X_2 \\ X_2 & X_3 \end{pmatrix} \quad \text{and the LMI becomes} \quad \sum_{j=0}^{3} L_j X_j \succeq 0$$

where the 4×4 symmetric matrices L_0, L_1, L_2, L_3 are:

$$L_0 := \begin{pmatrix} C^T C & 0 \\ 0 & I \end{pmatrix} \qquad L_1 := \begin{pmatrix} 2a_{11} & a_{21} & b_{11} & b_{12} \\ a_{21} & 0 & 0 & 0 \\ b_{11} & 0 & 0 & 0 \\ b_{12} & 0 & 0 & 0 \end{pmatrix}$$

$$L_2 := \begin{pmatrix} 2a_{12} & a_{11} + a_{22} & b_{21} & b_{22} \\ a_{22} + a_{11} & 2a_{21} & b_{11} & b_{12} \\ b_{21} & b_{11} & 0 & 0 \\ b_{22} & b_{12} & 0 & 0 \end{pmatrix} \qquad L_3 := \begin{pmatrix} 0 & 0 & 0 & a_{12} \\ 0 & 0 & 0 & a_{22} \\ 0 & 0 & 0 & b_{21} \\ a_{12} & a_{22} & b_{21} & 2b_{22} \end{pmatrix}$$

$+$ and $-$ of Keeping Matrices Whole

$+$ **not many variables**

$+$ **SHORT FORMULAS**

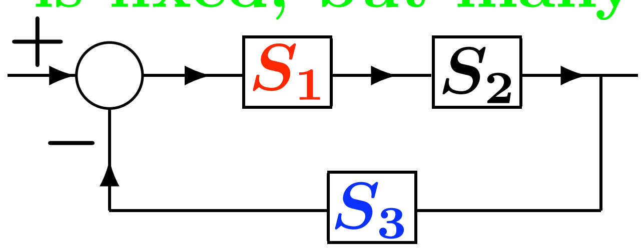
$-$ **Trouble is formulas are noncommutative.**

$+$ **NCAlgebra package does symbolic noncommutative algebra.**

DIMENSION FREE

1. **"OUTER" FREEDOM - NOT THIS TALK - Have "one" type of system and protocols for joining them together to form large networks without destroying stability. "One" system many topologies.**
   – **Kelly, Beallieu, Vinnicombe, Low, Paganini, Doyle, Basar**

2. **INNER FREEDOM – THIS TALK**
   Topology is fixed; but many systems . E.g.



WANT FORMULAS: which hold regardless of the dimension of system S_1, S_2, S_3. E.g. almost all control text problems.

Question: WHAT ARE THEIR DUALS?

SAMPLE PROBLEM: CONVEXITY

CONVEXITY IS VERY IMPORTANT FOR LINEAR SYSTEMS PROBLEMS

SINCE THEIR SOLNS ARE NUMERICAL OPTIMIZATION BASED

AND ONE NEEDS CERTAINTY OF HAVING A GLOBAL MINIMUM

CONVEXITY when

DIM FREE vs DIM DEPENDENT

THE RANGE OF QUESTIONS

Characterize and analyze these situations:

	LMI	*CONVEX*
DIMENSIONLESS(NC)	*are convex*	?
DIMENSION DEP.(COM)	?	*usual convex*

Q? Can we treat many more problems with convex techniques than LMI techniques?

Ultimately we want Symbolic Algorithms

NC CONVEX POLYNOMIALS

Function p of noncommutative variables $\vec{x} := (x_1, x_2)$ is MATRIX CONVEX (geometric def.) $0 \le \alpha \le 1$

$$p(\alpha \vec{X} + (1-\alpha)\vec{Y}) \preceq \alpha p(\vec{X}) + (1-\alpha)p(\vec{Y})$$

$$\frac{1}{2}p(\vec{X}) + \frac{1}{2}p(\vec{Y}) - p(\frac{1}{2}\vec{X} + \frac{1}{2}\vec{Y}) is\ Pos\ Def?$$

Question: Consider the noncommutative polynomial

$$p(x) := x^4 + (x^4)^T.$$

Is it matrix convex?

CONVEX POLYNOMIALS ARE TRIVIAL

THM: (McC + H)

Every symmetric polynomial in noncommutative variables $x_1, \cdots, x_g$ which is matrix convex (on any NC open set) has degree 2 or less.

One Var. Convex & Monotone:

. F. Hansen, M. Uchiyama, Jun Tomiyama

COR A Convex NC Polys is the Schur complement of some linear pencil.

Proof if convex everywhere:

1. **NC Positive Polynomials**
2. **NC Second Derivatives**
3. **Put the two together**

Convexity Algorithm

$$\textbf{In[1]} := \quad << \texttt{NCAlgebra.m;}$$

$$\textbf{In[2]} := \quad << \texttt{NCConvexity.m;}$$

$$\textbf{In[3]} := \quad \texttt{SetNonCommutative[X, Y];}$$

$$\textbf{In[4]} := \quad \texttt{F} \; = \; \texttt{inv[X} - \texttt{inv[Y]]}$$

$$\textbf{Out[5]} := \quad \textbf{inv[X} - \textbf{inv[Y]]}$$

$$\textbf{In[6]} := \quad \texttt{NCConvexityRegion[F,\{X, Y\}]}$$

$$\textbf{Out[7]} := \quad \{\{\texttt{2inv[X} - \texttt{inv[Y]]}, \; \texttt{2inv[Y]}\}\}$$

Download NCAlgebra : www.math.ucsd ∼ ncalg

NC RATIONAL

THEOREM (McCullough Vinnikov +H)

Suppose r is a symmetric noncommutative rational function in symmetric variables $\vec{x}$ which is matrix convex near 0 THEN

r is matrix convex on all of (the 0 component of) its domain.

AND

r has a representation in terms of an an LMI.

MORAL:

r CONVEX IN AN NC OPEN SET

OFTEN IMPLIES

r CONVEX EVERYWHERE

STATUS OF QUESTIONS

For Rational Functions

	LMI	*CONVEX*
$DIMLESS(NC)$	$CONVEX =$	$equivToLMI$
$DIM\ DEP.(COM)$	$n = 2, n > 2?$	$usual\ convexity$

Q? Can we treat more problems with convex techniques than LMI techniques?

CONJ. FOR DIMENSIONLESS SYSTEMS PROBLEMS CONVEX $\Leftrightarrow$ LMI

This is true under the restrictions of our set up.

BEYOND CONVEXITY

NC POLYS WHOSE 2^{nd} DERIVATIVE WHILE NOT POSITIVE ARE HIGHLY CONSTRAINED

with Dym and McCullough and Greene

sliHessSig.tex

HOW POSITIVE ARE YOU?

For f an NC symmetric polynomial in NC variables $\vec{x}$ we can represent f as a Sum and Difference of Squares (SDS)

$$f(\vec{x}) = \sum_{j=1}^{\sigma_+} f_j^+(\vec{x})^T f_j^+(\vec{x}) - \sum_{\ell=1}^{\sigma_-} f_\ell^-(\vec{x})^T f_\ell^-(\vec{x})$$

where f_j^+, f_j^- are NC polynomials.

f =SDS is highly non-unique.

$\sigma_\pm^{min}(f) :=$ the smallest number of positive (resp. negative squares) required in an SDS decomposition of f.

POSITIVELY POSITIVE $\sigma_-^{min} = 0$

$$f(\vec{x}) = \sum_{j=1}^{\sigma_+} f_j^+(\vec{x})^T f_j^+(\vec{x}) - \sum_{\ell=1}^{\sigma_-^{min}} f_\ell^-(\vec{x})^T f_\ell^-(\vec{x})$$

EXAMPLE: **matrix positive** **NC symmetric polynomials, namely f with the property that for each tuple $\vec{X} = \{X_1, X_2, \ldots X_g\}$ of $n \times n$ symmetric matricies X_j the matrix $f(\vec{X})$ is positive semidefinite matrix.**

$\sigma_-^{min}(f) = 0$ is equivalent to $f(\vec{X})$ matrix positive.

THM: **(McCullough, H) $p''(\vec{X})[\vec{H}]$ is positive semidefinite for all $\vec{X}, \vec{H} \implies \deg p \leq 2$.**

that is CONVEX p HAVE DEGREE TWO.

NONCOMMUTATIVE NONCONVEXITY

BEYOND CONVEXITY:

p with $p''(\vec{x})[\vec{h}]$ having prescribed "negativity"?

$$p''(\vec{x})[\vec{h}] = \sum_{j=1}^{\sigma_+} f_j^+(\vec{x})^T[\vec{h}]f_j^+(\vec{x})[\vec{h}] - \sum_{\ell=1}^{\sigma_-^{min}} f_\ell^-(\vec{x})^T[\vec{h}]f_\ell^-(\vec{x})[\vec{h}]$$

THM: (Dym, McCullough, H) SUPPOSE $p(\vec{x})$ is a symmetric polynomial in NC symmetric variables $x_1, \ldots, x_g$, THEN

$$degree(p) \le 2\sigma_\pm^{min}(p'') + 2.$$

Ex: Says for $\sigma_\pm^{min}(p'') = 1$, that $degree(p) \le 4$.

FINER STRUCTURE

THM: (Dym, Greene, McCullough, H) SUPPOSE

$$\sigma_{-}^{min}(p'') = 1,$$

THEN $d \le 4$ **and** $p = LFL + LQ + Q^T L + B$,

If $d = 3$**, then** L, F **have degree 1 and** q **have degree 2.**

If $degree(p) = 4$**, then** L **is linear,** Q, F **homogeneous of degree 2, and** B **have degree 2.**

LOCAL STRUCTURE implies GLOBAL

NC FUNCTION *F* HAS

STRUCTURE IN AN NC OPEN SET

OFTEN IMPLIES

***F* HAS THE STRUCTURE EVERYWHERE**

EXAMPLES:

Polynomials with “curvature” of given NC signature.

Convex Rational Functions

THE FUTURE AND NC CONVEXITY

FINISH PROVING (OR REFUTING) THAT NC CONVEXITY IMPLIES LMI

NC CHANGE OF VARIABLES TO MAKE CONVEX

NC SYMBOLIC NUMERICAL ALGORITHMS

NEW TOPIC

PROOFS AND FREE DIMENSION

NC POSSTELLENSATZ

NO RELAXATION NEEDED:

THM: [H 2002]

Matrix positive non-commutative polynomials are sums of squares

$$p = SoS$$

Do with Putinar's famous proof.

NC POSSTELLENSATZ

THM: [Commutative Real NullSS]

If q is zero where p is zero, then

$$q^{2n} + SoS = rp$$

FOR SOME NC NO RELAXATION NEEDED:

THM: [NC Real NullSS: H-McCullough Putinar]

Suppose p contains no x^T. If $q(X)v$ is zero wherever $p(X)v = 0$ is zero, then $q = rp$

Do with Putinar's famous proof plus dilation argument.

If p contains transposes, NullSS may be false.

SOME NC PROOFS

Given a set of noncommutative polynomials $p_1(x), \cdots, p_k(x)$ in variables

$$x_1 < x_2 < \cdots < x_k$$

ordered according to our dislike for them; we want to eliminate x_k the most.

A NONCOMMUTATIVE GRÖBNER BASIS

$$q_1(x), q_2(x), q_3(x), \cdots$$

Has Properties:

1. **if set to 0 has the same set of solutions as the p.**
2. **is in as "triangular form" as possible.**

For example, $q_1(x_1)$ - depends only on x_1

$q_1(x_1, x_2)$ - depends only on x_1, x_2

NC PROOF

ALGORITHM (Mora) idea is simple:
Pick p_i, p_j; cross multiply and eliminate highest order terms to produce a new polynomial. Keep going.

ALGORITHM (Mora) $\sim$ Knuth-Bendix Algorithm?
(a rumor i heard)

My Game was: Discover classical theorems in linear system theory fairly automatically.

OUR EXPERIMENTS

Book Zu Doyle Glover = linear control up to 1990.

“CONCLUDE”:

1. about $\frac{1}{2}$ of the theorems can be ”discovered” by an NC GB runs

2. All but one Theorem can be discovered” by an NC GB run plus a *single* change of variable

Single change of variable **means:**

keep all variables fixed except one, change it.

¿CAN WE AGGREGATE?

MATH QUESTION:

Given 25 polynomials in 50 variables $p_1, \cdots, p_{25}$.

Is there a single NC polynomial $p(x, y)$ such that when p is applied to the 5×5 matrices

$$X = \{X_{ij}\}_{i=1,\cdots 5}, \; Y = \{Y_{ij}\}_{i=1,\cdots 5}$$

the 5×5 matrix valued polynomial contains the given p_i.

SHORTER FORMULAS MEAN SHORTER PROOFS?

Suppose one has list of (commutative) polynomial equations $p_j(x) = 0$ in 50 variables
One wants to PROVE these are equivalent to the simpler set $q_j(x) = 0$ of equations.

We do this by eliminating variables with a GB algorithm

Suppose secretly the p_j aggregate to two NC polynomials P^1, P^2 in two variables. We want to find nice Q_j.

We do this by eliminating variables with an NC GB algorithm

¿WHICH PROOF IS SHORTER?

My bet is aggregation will often be better in practice.

¿ NONLINEAR FREEDOM ?

SEARCH FOR THE BETTER
MATH QUESTION

IS THERE A NICE CLASS OF NONLINEAR DIMENSION FREE SYSTEMS TO STUDY?

GIVEN SYSTEM LAYOUT WHAT CLASS OF NONLINEARITIES IS INTERESTING, BUT DOES NOT RULE OUT ALL STRUCTURE.

(not optimistic)

END

ALGORITHMS and IMPLEMENTATIONS

1. **Convexity Checker - Camino, Skelton, H Public**

2. **Realization Builder: Convex Rational to LMI - Slinglend, Shopple in progress**

3. **Numerical matrix unknowns - Camino, Skelton, H in house**

4. **LMI Producer (uses existing methods on special problems) de Oliveira, H in house (out soon)**

Try NCAlgebra