Optimization in Engineering

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Introduction

Optimization! What to solve? How to solve?

So far...

- H_∞ analysis (Control theory), [CDC'13], [MSC'14]
- Weighted sum rate maximization (Communication) [IEEE/ACM TON]
- Frequency control in power grid (Power engineering) [CDC'14]
- Fast consensus protocol for directed graphs (Control theory) [ACC'14]
- Optimal delay distribution design (Control theory) [MSC'14]
- fMRI connectivity (Neuroscience) [????]

Lessons learned..

- Interesting problems are often non-convex.
  - Good heuristic *exists*
- Similar patterns
  - Lifting and Projection
  - Formulation actually matters
  - Naive formulation ⇒ Convex ⇒ non-Convex ⇒ Good heuristic
-

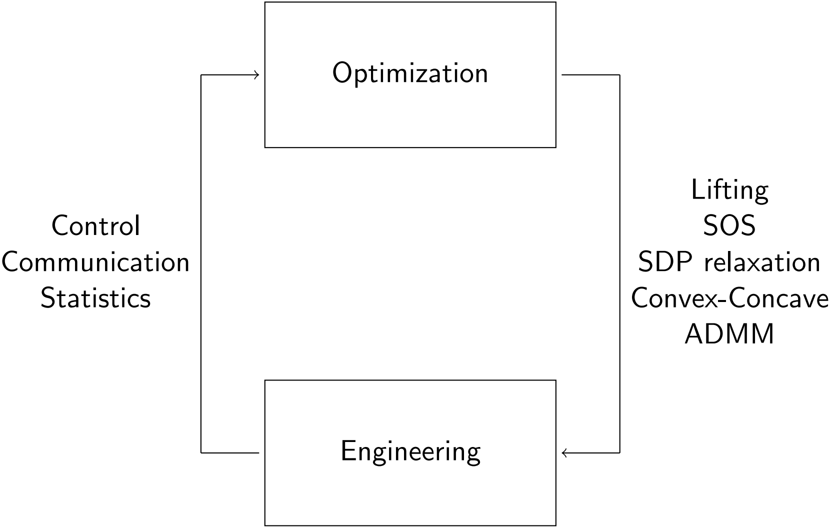
Lessons learned..

- Interesting problems are often non-convex.
  - Good heuristic *exists*

- Similar patterns
  - Lifting and Projection
  - Formulation actually matters
  - Naive formulation $\Rightarrow$ Convex $\Rightarrow$ non-Convex $\Rightarrow$ Good heuristic

- Optimization engineering in engineering!
  - Short term: Catalog of common patterns.
  - Long term: Unified theory?????

Optimization engineering in engineering



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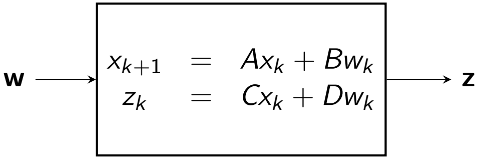
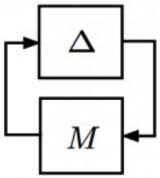
H_∞ Analysis

H_∞ Norm



- $\|\mathcal{M}\|_\infty = \sup_{\mathbf{w}} \frac{\|\mathbf{z}\|_2}{\|\mathbf{w}\|_2}$.
- The disturbance **w** tries to maximize the output **z**! (l_2 induced gain)

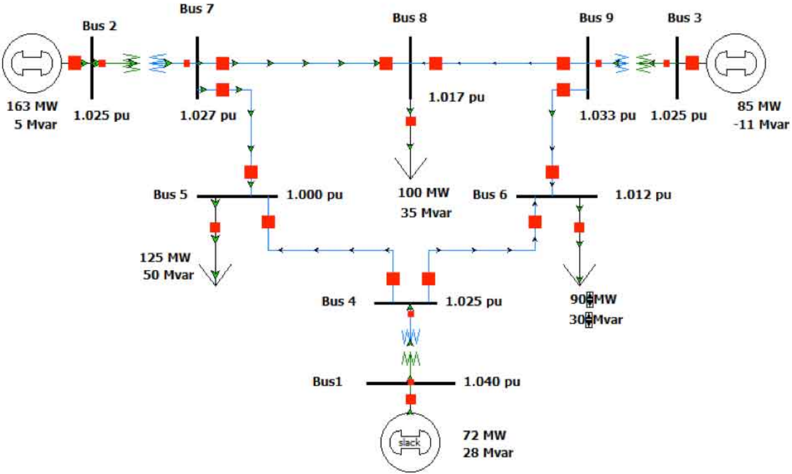
H_∞ Norm



- $\|M\|_\infty = \sup \frac{\|\mathbf{z}\|_2}{\|\mathbf{w}\|_2}$
- Robust Stability: $||\Delta M|| = ||\Delta|| \cdot ||M|| < 1!$
- H_∞ Norm characterizes an allowable size of disturbances: *The smaller the better*

H_∞ Norm

Power System Analysis.



H_∞ Norm

- Use power norm instead of l_2 norm[1]:

$$\|\mathbf{h}\|_p^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h_k^* h_k.$$

- System $\mathcal{M}$:

$$\begin{aligned} x_{k+1} &= Ax_k + Bw_k \\ z_k &= Cx_k + Dw_k. \end{aligned}$$

- H_∞ norm:

$$\|\mathcal{M}\|_\infty = \sup_{\|\mathbf{w}\|_p \le 1} \frac{\|\mathbf{z}\|_p}{\|\mathbf{w}\|_p} = \sup_{\|\mathbf{w}\|_p \le 1} \|\mathbf{z}\|_p.$$

  - $\|\mathcal{M}\|_\infty < \infty$ iff A is *stable*.

[1] Zhou, Kemin, et al. "Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives. I. Robust performance analysis." Automatic Control, IEEE Transactions on 39.8 (1994): 1564-1574.

H_∞ Norm

- Textbook approach[2]

$$\begin{aligned} \underset{\lambda, P}{\text{minimize}} \quad & \lambda \\ \text{subject to} \quad & \begin{bmatrix} A^*PA - P & A^*PB \\ B^*PA & B^*PB \end{bmatrix} + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - \lambda I \end{bmatrix} \preceq 0 \\ & \lambda \geq 0, P = P^*. \end{aligned}$$

  - $\|\mathcal{M}\|_\infty^2 = \lambda_{\text{opt}}$.
  - Based on Kalman-Yakubovich-Popov lemma.
  - Require controllability of (A, B).
  - More importantly, WHAT IS THIS???
  -

[2]Dullerud, Geir E., and Fernando Paganini. A course in robust control theory. Vol. 6. New York: Springer, 2000.

H_∞ Norm

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  - Based on Kalman-Yakubovich-Popov lemma.
  - Require controllability of (A, B).
  - More importantly, WHAT IS THIS???
  - Dual side, not the Primal side

[2] Dullerud, Geir E., and Fernando Paganini. A course in robust control theory. Vol. 6. New York: Springer, 2000.

Direct Approach

$$
\begin{aligned}
||\mathcal{M}||_\infty^2 = \underset{\mathbf{w},\mathbf{x}}{\text{maximize}} \quad & \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} z_k^* z_k \\
\text{subject to} \quad & x_{k+1} = Ax_k + Bw_k \\
& z_k = Cx_k + Dw_k \\
& x_0 = 0 \\
& ||\mathbf{w}||_p \leq 1.
\end{aligned}
$$

- Infinite-dimensional non-convex optimization.
-

Direct Approach

$$
\begin{aligned}
||\mathcal{M}||_\infty^2 = \underset{\mathbf{w},\mathbf{x}}{\text{maximize}} \quad & \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} z_k^* z_k \\
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& z_k = Cx_k + Dw_k \\
& x_0 = 0 \\
& ||\mathbf{w}||_p \leq 1.
\end{aligned}
$$

- Infinite-dimensional non-convex optimization.
- Trick: Lifting!

Lifting

Introduce "Covariance-like" Matrix, V:

$$V = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix}^* = \begin{bmatrix} < xx^* > & < xw^* > \\ < wx^* > & < ww^* > \end{bmatrix} \succeq 0.$$

- Objective:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} z_k^* z_k = \operatorname{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix}^* \right)$$

- Constraint:

$$\begin{aligned} x_{k+1}x_{k+1}^* &= (Ax_k + Bw_k)(Ax_k + Bw_k)^* \\ \Rightarrow \quad \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} &= \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix}. \end{aligned}$$

Lifting

- Lifted problem:

$$
\begin{aligned}
||\mathcal{M}||_\infty^2 = \ \underset{V,\mathbf{x},\mathbf{w}}{\text{maximize}} \quad & \|\mathbf{z}\|_p^2 = \operatorname{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix}^*\right) \\
\text{subject to} \quad & \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix} \\
& \operatorname{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) \leq 1 \\
& V \succeq 0 \\
& x_{k+1} = Ax_k + Bw_k \\
& x_0 = 0, \|\mathbf{w}\|_p \leq 1 \\
& V = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix}^*
\end{aligned}
$$

  - Have NOT changed the problem yet.
  -

Lifting

- Lifted problem:

$$
\begin{aligned}
||\mathcal{M}||_\infty^2 = \underset{V,\mathbf{x},\mathbf{w}}{\text{maximize}} \quad & \|\mathbf{z}\|_p^2 = \operatorname{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix}^*\right) \\
\text{subject to} \quad & \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix} \\
& \operatorname{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) \leq 1 \\
& V \succeq 0 \\
& x_{k+1} = Ax_k + Bw_k \\
& x_0 = 0, \|\mathbf{w}\|_p \leq 1 \\
& V = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix}^*
\end{aligned}
$$

  - Have NOT changed the problem yet.
  - Semidefinite program!

Lifting and Projection

- Lifted problem:

$$\begin{aligned}
\mu_{\text{opt}} = \ \underset{V}{\text{maximize}} \quad & \operatorname{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix}^*\right) \\
\text{subject to} \quad & \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix} \\
& \operatorname{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) \leq 1 \\
& V \succeq 0
\end{aligned}$$

  - $\mu_{\text{opt}} \geq \|\mathcal{M}\|_\infty^2$, in fact, $\mu_{\text{opt}} = \|\mathcal{M}\|_\infty^2$.
  - Moreover, V_{opt} gives a solution to $\|\mathcal{M}\|_\infty^2$!

Proof Idea

1. The extreme points of the feasible set are *Rank one* matrices.[3]
2. $\text{Rank}(V_{\text{opt}}) = 1$ since objective is *linear*. (If not we can construct it)
3. $V_{\text{opt}} = \begin{bmatrix} x_{\text{opt}} \\ w_{\text{opt}} \end{bmatrix} \begin{bmatrix} x_{\text{opt}} \\ w_{\text{opt}} \end{bmatrix}^*.$
4. $e^{j\theta_{\text{opt}}} x_{\text{opt}} = A x_{\text{opt}} + B w_{\text{opt}}.$
5. Worst case disturbance: $w_k = e^{j\theta_{\text{opt}} k} w_{\text{opt}}!$

[3] Rantzer, Anders. "On the Kalman–Yakubovich–Popov lemma." Systems & Control Letters 28.1 (1996): 7-10.

H_∞ analysis: Primal problem

- Primal problem:

$$
\begin{aligned}
\mu_{\text{opt}} = \underset{V}{\text{maximize}} \quad & \operatorname{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix}^*\right) \\
\text{subject to} \quad & \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix} \\
& \operatorname{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) \leq 1 \\
& V \succeq 0
\end{aligned}
$$

  - No need for the controllability (A, B).
  - Worst case disturbance can be extracted.

H_∞ analysis: Dual problem

- Dual problem:

$$\begin{aligned} \underset{\lambda, P}{\text{minimize}} \quad & \lambda \\ \text{subject to} \quad & \begin{bmatrix} A^*PA - P & A^*PB \\ B^*PA & B^*PB \end{bmatrix} + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - \lambda I \end{bmatrix} \preceq 0 \\ & \lambda \geq 0, P = P^*. \end{aligned}$$

  - It's KYP![4]
  - Strong duality?
  - Worst case disturbance $\mathbf{w}$?

---

[4] V. A. Yakubovich, Solution of certain matrix inequalities encountered in non-linear regulation theory, Doklady Akademii Nauk SSSR, vol. 143, pp. 13041307, 1962

H_∞ analysis: Dual problem

- Dual problem:

$$
\begin{aligned}
\underset{\lambda, P}{\text{minimize}} \quad & \lambda \\
\text{subject to} \quad & \begin{bmatrix} A^*PA - P & A^*PB \\ B^*PA & B^*PB \end{bmatrix} + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - \lambda I \end{bmatrix} \preceq 0 \\
& \lambda \geq 0, P = P^*.
\end{aligned}
$$

  - It's KYP![4]
  - Strong duality? Controllability of (A, B)
  - Worst case disturbance $\mathbf{w}$? No

[4] V. A. Yakubovich, Solution of certain matrix inequalities encountered in non-linear regulation theory, Doklady Akademii Nauk SSSR, vol. 143, pp. 13041307, 1962

H_∞ analysis: Dual problem

Counter example: $(A, B, C, D) = (0, 0, 1, 1)$.

- Primal problem: 1

$$\begin{aligned} \underset{V}{\text{maximize}} \quad & \operatorname{Tr}\left(\begin{bmatrix}1 & 1\end{bmatrix} V \begin{bmatrix}1 & 1\end{bmatrix}^*\right) \\ \text{subject to} \quad & \begin{bmatrix}1 & 0\end{bmatrix} V \begin{bmatrix}1 \\ 0\end{bmatrix} = 0 \\ & \operatorname{Tr}\left(\begin{bmatrix}0 & 1\end{bmatrix} V \begin{bmatrix}0 & 1\end{bmatrix}^*\right) \leq 1 \\ & V \succeq 0 \end{aligned}$$

- Dual problem: $+\infty$

$$\begin{aligned} \underset{\lambda, P}{\text{minimize}} \quad & \lambda \\ \text{subject to} \quad & \begin{bmatrix}1 & 1 \\ 1 & 1 - \lambda I\end{bmatrix} \preceq 0 \\ & \lambda \geq 0. \end{aligned}$$

Extension to bounded frequency H_∞ analysis

$$
\begin{aligned}
||\mathcal{M}||_{\mathcal{L}}^2 = \underset{\mathbf{w},\mathbf{x}}{\text{maximize}} \quad & \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} z_k^* z_k \\
\text{subject to} \quad & x_{k+1} = Ax_k + Bw_k \\
& z_k = Cx_k + Dw_k \\
& x_0 = 0 \\
& \mathbf{w} \in \mathcal{W}_L
\end{aligned}
$$

- $\mathcal{W}_L = \{\mathbf{w} : w_k = e^{j\theta k} w_s,\ \theta \in [-\theta_0, \theta_0]\}$.
- Low frequency disturbance.

Extension to bounded frequency H_∞ analysis

- Same lifting approach:

$$
\begin{aligned}
\underset{V}{\text{maximize}} \quad & \operatorname{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix}^*\right) \\
\text{subject to} \quad & \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix} \\
& \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix} \\
& \succeq 2\cos\theta_0 \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} \\
& \operatorname{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) \leq 1 \\
& V \succeq 0.
\end{aligned}
$$

  - Extreme points are Rank one matrices.
  - Same argument.

Extension to bounded frequency H_∞ analysis

- Dual problem:

$$
\begin{aligned}
\underset{\lambda,P}{\text{minimize}} \quad & \lambda \\
\text{subject to} \quad & \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} P & Q \\ Q & -P - 2\cos\theta_0 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \\
& + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} \preceq \lambda \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \\
& \lambda \geq 0, P = P^*, Q \succeq 0,
\end{aligned}
$$

  - It's the Generalized KYP! [5]
  - Strong duality holds when (A, B) is controllable.
  - No way to extract worst case disturbance.

[5] Iwasaki, Tetsuya, and Shinji Hara. "Generalized KYP lemma: unified frequency domain inequalities with design applications." Automatic Control, IEEE Transactions on 50.1 (2005): 41-59.

Extension to bounded frequency H_∞ analysis

- High frequency disturbance:

$$
\begin{aligned}
\underset{V}{\text{maximize}} \quad & \mathrm{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix}^*\right) \\
\text{subject to} \quad & \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix} \\
& \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix} \\
& \preceq 2\cos\theta_0 \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} \\
& \mathrm{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) \leq 1 \\
& V \succeq 0.
\end{aligned}
$$

Summary

- Direct approach.
- No need for controllability.
- Worst case disturbance extraction.
- Extension to bounded frequency disturbances.
- The derivations with continuous time dynamics, descriptor systems are similar.
-

Summary

- Direct approach.
- No need for controllability.
- Worst case disturbance extraction.
- Extension to bounded frequency disturbances.
- The derivations with continuous time dynamics, descriptor systems are similar.
- More clean picture!

Future Directions

- Specialized solver: Connection to Algebraic Riccati equation, Hamiltonian based approach.
  - Stephen Boyd, and Venkataramanan Balakrishnan. "A regularity result for the singular values of a transfer matrix and a quadratically convergent algorithm for computing its L_∞-norm." Systems & Control Letters 15.1 (1990): 1-7.
  - Pablo A Parrilo, "On the numerical solution of LMIs derived from the KYP lemma." Decision and Control, 1999. Proceedings of the 38th IEEE Conference on. Vol. 3. IEEE, 1999.
  - Zhang Liu, and Lieven Vandenberghe. "Low-rank structure in semidefinite programs derived from the KYP lemma." Decision and Control, 2007 46th IEEE Conference on. IEEE, 2007.

Future Directions

- Specialized solver: Connection to Algebraic Riccati equation, Hamiltonian based approach.
- Redo H_∞ synthesis
  - Minimax formulation.
  - Heuristic for structured controller ($u = Kx$), minimax iteration..

$$
\begin{aligned}
\underset{K}{\text{minimize}}\ \underset{V}{\text{maximize}} \quad & \operatorname{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix}^*\right) \\
\text{subject to} \quad & \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} \\
& \succeq \begin{bmatrix} (A + B_1 K) & B_2 \end{bmatrix} V \begin{bmatrix} (A + B_1 K)^* \\ B_2^* \end{bmatrix} \\
& \operatorname{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) \leq 1 \\
& V \succeq 0
\end{aligned}
$$

Future Directions

- Specialized solver: Connection to Algebraic Riccati equation, Hamiltonian based approach.
- Redo H_∞ synthesis
  - Minimax formulation.
  - Heuristic for structured controller ($u = Kx$), minimax iteration..
- Scalable H_∞ analysis?
  - $V \succeq 0$: Not scalable.
  - Diagonally dominant V (LP), $V_{ii}V_{jj} \geq V_{ij}^2$ (SOCP).
- Application
  - Robust analysis of power grid.
  - Robustness of a graph?

Conclusion

- Time to look back some classical problems.
- More on interesting optimization problems.
- Optimization engineering??