Laplace Transform

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1 Introduction

Consider a differential equation with *constant coefficients* as:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 x = b_m u^m + b_{m-1} u^{m-1} + \ldots + b_0 u \tag{1}$$

Suppose the above equation is a model for a dynamical system of interest: the solution $x(t)$ is then the state of such system, while $u(t)$ is its input. Furthermore assume that $n > m$: this is equivalent to assuming that our system is *causal* (the state at time t only depends on past states and inputs).
We can easily find the solution to a differential equation of this type using the *Laplace transform*.

2 Definition

Under suitable assumptions, the Laplace Transform (LT) is a bijective (and therefore invertible) mapping that links functions of time $f(t)$ with functions of complex argument $F(s)$, $s = \sigma + j\omega \in \mathbb{C}$. A key feature of this transformation is that of mapping differential equations to algebraic equations: in particular, the convolution operation will correspond to the multiplication operation, as we will see in Section 3.

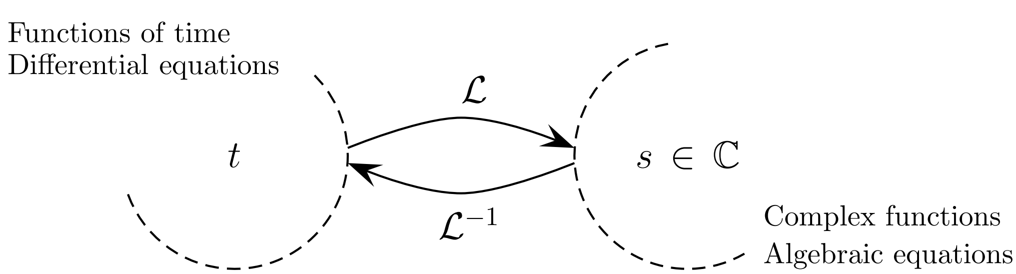


Figure 1: Laplace Transform mapping

The Laplace Transform of a function $f(t)$, piecewise continuous and bounded such that

$f(t) \neq 0$ for $t \geq 0$, is defined [1] as:

$$F(s) = \mathcal{L}[f(t)] \triangleq \int_{0^-}^{+\infty} f(t)e^{-st}dt \tag{2}$$

For those values of s such that the integral converges.
It can be proved that if the integral converges for s_0, then it will for all s such that $\mathcal{R}e\{s\} > \mathcal{R}e\{s_0\} = \sigma_0$. The value σ_0 limits the *region of convergence*[2], or ROC, of the transform.

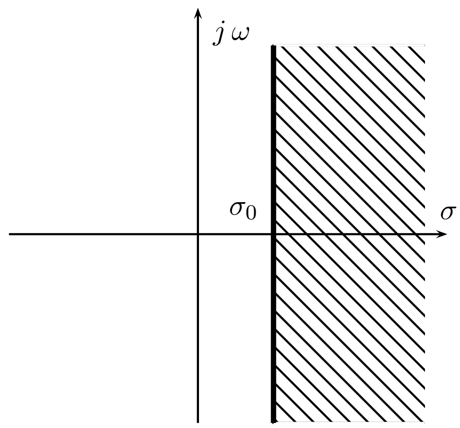


Figure 2: ROC in the complex plane

3 Properties of the Laplace Transform

- **Linearity** The integral operation is linear, therefore the LT inherits such property:

$$\mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1\mathcal{L}[f_1(t)] + c_2\mathcal{L}[f_2(t)]$$

- **Transformation of the integral** If functions $f(t)$, $g(t) = \int_{0^-}^{t} f(\tau)d\tau$ have LT with the same ROC:

$$\mathcal{L}[g(t)] = \frac{1}{s}\mathcal{L}[f(t)]$$

- **Trasformation of the derivative** If $f(t)$, $f'(t)$ have LT with the same ROC:

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0^-)$$

  This can be extended to the derivative of order n:

$$\mathcal{L}[f^n(t)] = s^n\mathcal{L}[f(t)] - \sum_{k=0}^{n-1} s^{n-k-1}f^k(0^-)$$

[1]This definition allows to consider functions that present an impulse at the origin. The bilateral Laplace Transform runs the integral from $-\infty$ to $+\infty$

[2]More precisely:

$$\sigma_0 \triangleq inf\left\{\mathcal{R}e\{s\},\, s \in \mathbb{C} \,:\, \int_{0^-}^{+\infty} f(t)e^{-st}dt = K < \infty\right\}$$

- **Frequency shift** If $f(t)$ admits a LT and $k \in \mathbb{C}$, the function $e^{kt}f(t)$ has ROC $\sigma_0 + \mathcal{R}e\{k\}$ and:

$$\mathcal{L}[e^{kt}f(t)] = F(s-k)$$

In fact:

$$\int_{0^-}^{+\infty} e^{kt}f(t)e^{-st}dt = \int_{0^-}^{+\infty} f(t)e^{-(s-k)t}dt$$

- **Time shift** If $f(t)$ admits a LT, the introduction of a time shift $t_0 > 0$ still yields a LT with the same ROC and :

$$\mathcal{L}[f(t-t_0)\cdot 1(t-t_0)] = e^{-st_0}F(s)$$

This can be shown with a change of variable:

$$\int_{0^-}^{+\infty} f(t-t_0)e^{-st}dt = \int_{-t_0}^{+\infty} f(\tau)e^{-s\tau}e^{-st_0}dt = e^{-st_0}F(s)$$

- **Multiplication by t^n** Assume $n = 1$, and develop the derivative of $F(s)$:

$$\begin{aligned}\frac{d}{ds}F(s) &= \frac{d}{ds}\int_{0^-}^{+\infty} f(t)e^{-st}dt = \int_{0^-}^{+\infty} f(t)\frac{d}{ds}e^{-st}dt \\ &= -\int_{0^-}^{+\infty} f(t)te^{-st}dt = -\int_{0^-}^{+\infty} tf(t)e^{-st}dt = -\mathcal{L}[tf(t)]\end{aligned}$$

This yields the general formula:

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$$

- **Time scaling**

$$\mathcal{L}[f(at)] = \frac{1}{a}F(\frac{s}{a}),\ a \in \mathbb{R}^+$$

Again this can be proved with a change of variables:

$$\int_{0^-}^{+\infty} f(at)e^{-st}dt = \int_{0^-}^{+\infty} f(t)e^{-s\frac{\tau}{a}}d(\frac{\tau}{a}) = \frac{1}{a}F(\frac{s}{a})$$

- **Convolution** This is probably the most important property of the LT. Recall the definition of *convolution integral*: given $f(t)$, $g(t)$, $f(t) = g(t) = 0$ fpr $t \leq 0^-$,

$$h(t) \triangleq f(t) \circledast g(t) = \int_{0^-}^{t} f(\tau)g(t-\tau)d\tau$$

It can be proved that:

$$\mathcal{L}[f(t) \circledast g(t)] = F(s)\cdot G(s)$$

In fact:

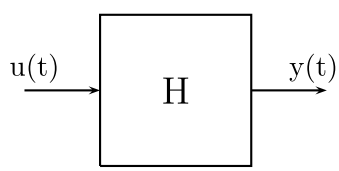
$$
\begin{aligned}
\int_{0^-}^{+\infty} [f(at)f(t) \circledast g(t)]dt &= \int_{0^-}^{+\infty} \int_{0^-}^{t} f(\tau)g(t-\tau)d\tau e^{-st}dt \\
&= \int\int_{\mathcal{D}} f(\tau)g(t-\tau)e^{-st}d\tau dt
\end{aligned}
$$

Where $\mathcal{D} = \{(t,\tau) \in \mathbb{R}^2 \,|\, \tau \leq t\}$ is the domain of integration. This means that:

$$
\begin{aligned}
\int_{0^-}^{+\infty} [f(at)f(t) \circledast g(t)]dt &= \int_{0^-}^{+\infty} \{\int_{\tau}^{\infty} f(\tau)g(t-\tau)e^{-st}dt\}d\tau \\
&= \int_{0^-}^{+\infty} \int_{0^-}^{+\infty} e^{s(\tau+\sigma)} f(\tau)g(\sigma)d\tau d\sigma \\
&= F(s) \cdot G(s)
\end{aligned}
$$

Note: for $t > \tau$, $g(t-\tau) \equiv 0$.
This is a fundamental property, see Figure 3.

$$y(t) = \int_{0^-}^{t} h(t-\tau)u(\tau)d\tau$$



$$\Longrightarrow \quad Y(s) = H(s)U(s)$$

Figure 3: LT of a convolution integral

- **LT of a periodic function** Consider a periodic function $f(t)$ as the one represented in Figure 4 and defined as:

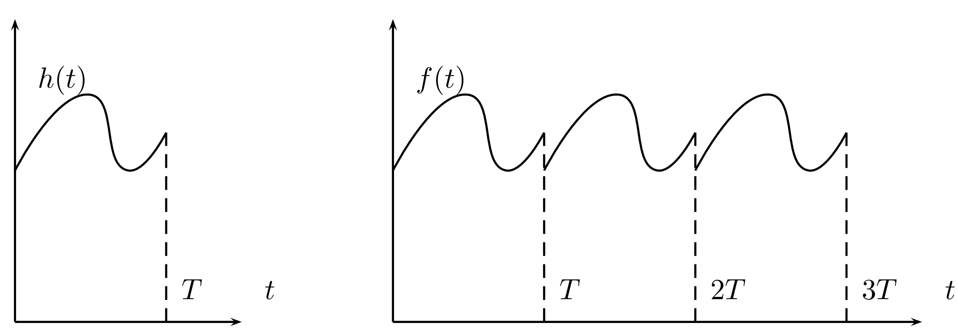


Figure 4: Periodic function

$$f(t) = \begin{cases} h(t) & for\, t \in [0^+, T]) \\ 0 & otherwise \end{cases}$$

The function $f(t)$ can also be described as $f(t) = \sum_{k=0}^{+\infty} h(t - kT)$. For the linearity and time shift properties:

$$F(s) = \mathcal{L}[f(t)] = \sum_{k=0}^{+\infty} H(s)e^{-skT} = F(s) \sum_{k=0}^{+\infty} e^{-skT} = \frac{H(s)}{1 - e^{-sT}}$$

Since $\sum_{k=0}^{+\infty} e^{-skT}$ is a geometric series.

4 List of transformations

- LT of the step function:

$$\mathcal{L}[1(t)] = \int_{0^-}^{+\infty} 1(t)e^{-st}dt = -\frac{1}{s}[e^{-st}]_0^{\infty} = \frac{1}{s} \qquad (\sigma_0 = 0)$$

- LT of the Dirach $\delta(t)$:

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{+\infty} \delta(t)e^{-st}dt = \int_{-\infty}^{+\infty} \delta(t)e^{-st}dt = 1 \qquad (\sigma_0 = -\infty)$$

Note: the expression of the LT of $\delta(t)$ allows to easily find the LT of a time series of impulses (Figure 5):

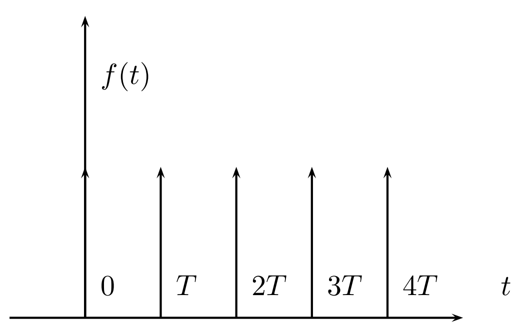


Figure 5: Series of impulses

$$\mathcal{L}[\sum_{k=0}^{+\infty} \delta(t - kT)] = \sum_{k=0}^{+\infty} e^{-skT} \cdot 1 = \frac{1}{1 - e^{-sT}}$$

This result is based on the transformation of a periodic function.
Recall the convolution integral property: multiplication in the complex domain implies convolution in the time domain. Then any periodic function $f(t)$, having period T, can be seen as the convolution of a "basis" function defined between 0 and T with a series of impulses having the same period.

- LT of the exponential function:

$$\mathcal{L}[e^{kt}1(t)] = \int_{0^-}^{+\infty} e^{kt}e^{-st}dt = \int_{0^-}^{+\infty} e^{(k-s)t}dt = \frac{1}{k-s}[e^{(k-s)t}]_0^{+\infty} = \frac{1}{s-k} \qquad (\sigma_0 = \mathcal{R}e\{k\})$$

- LT of t^n

$$\mathcal{L}[t^n 1(t)] = (-1)^n\{(-1)^n \frac{n!}{s^{n+1}}\} = \frac{n!}{s^{n+1}} \qquad (\sigma_0 = 0)$$

  This can be shown using the step function LT and the multiplication by t^n.

- LT of $t^n\, e^{kt}$:

$$\mathcal{L}[t^n\, e^{kt}\, 1(t)] = \frac{n!}{(s-k)^{n+1}}$$

- LT of $\sin(\omega t)$. Recall Euler formulas $e^{j\phi} = \cos\phi + j\,\sin\phi$, $e^{-j\phi} = \cos\phi - j\,\sin\phi$:

$$\mathcal{L}[\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \cdot 1(t)] = \frac{1}{2j}\left\{\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right\} = \frac{1}{2j}\frac{2j\omega}{s^2+\omega^2} = \frac{\omega}{s^2+\omega^2} \qquad (\sigma_0 = 0)$$

- LT of $\cos(\omega t)$:

$$\mathcal{L}[\frac{e^{j\omega t} + e^{-j\omega t}}{2} \cdot 1(t)] = \frac{1}{2}\left\{\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right\} = \frac{1}{2}\frac{2s}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} \qquad (\sigma_0 = 0)$$

5 Examples

Example 5.1 *Find the Laplace Transform of the function represented in Figure 6.*

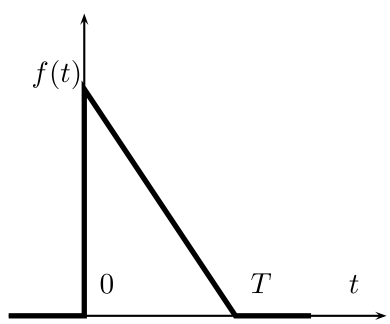


Figure 6: Example 1

We can see this function $f(t)$ as the sum of three elementary parts:

- A step at $t = 0$
- A ramp of slope $(-\frac{1}{T})$ at $t = 0$
- A ramp of slope $(+\frac{1}{T})$ at $t = T$

$$f(t) = 1(t) + (-\frac{1}{T})t + (\frac{1}{T})(t-T)$$

The linearity property of LT's allows immediately to find:

$$\mathcal{L}[f(t)] = \frac{1}{s} + (-\frac{1}{T})\frac{1}{s^2} + \frac{1}{T}[\frac{1}{s^2}e^{-Ts}] = \frac{1}{s} - \frac{1}{Ts^2}[1 - e^{-Ts}]$$

△

Example 5.2 *Find the Laplace Transform of the function given in Figure 7:*

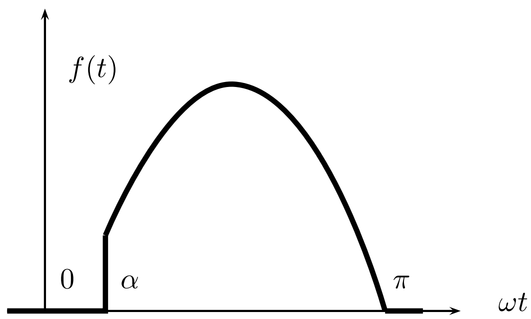


Figure 7: Example 2

The function of Figure 7 is just the delayed version of what shown in Figure 8.

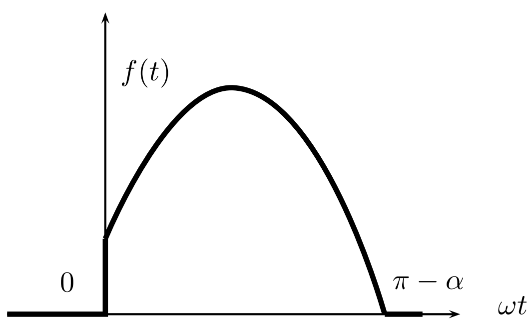


Figure 8: Example 2, function shifted to the origin

Moreover, $f_1(t)$ in Figure 8 can be interpreted as the difference of

- A sinusoid anticipated of $\frac{\alpha}{\omega}$ and zero for $t < 0$
- A sinusoid delayed of $\frac{(\pi-\alpha)}{\omega}$ and zero for $t < 0$

$$f_1(t) = \sin(\omega t + \alpha)\, 1(t) - \sin(\omega t - (\pi - \alpha))\, 1(t)$$

All half periods cancel except the first. We can now use the sinusoid angle addition formulas:

$$\sin(\omega t + \alpha) = \sin(\omega t)\cos\alpha + \cos(\omega t)\sin\alpha$$

and we can find:

$$\mathcal{L}[f_1(t)] = \frac{\omega \cdot \cos\alpha + s \cdot \sin\alpha}{s^2 + \omega^2} + \frac{\omega}{s^2 + \omega^2}\, e^{-\frac{\pi-\alpha}{\omega}}$$

Finally we can find the LT of $f(t)$, which is $f_1(t)$ with a delay of $\frac{\alpha}{\omega}$.

$$\mathcal{L}[f(t)] = \mathcal{L}[f_1(t)]\, e^{-\frac{\alpha}{\omega}} = \frac{\omega \cdot \cos\alpha + s \cdot \sin\alpha}{s^2+\omega^2}\, e^{-\frac{\alpha}{\omega}} + \frac{\omega}{s^2+\omega^2}\, e^{-\frac{\pi}{\omega}}$$

△

Example 5.3 Solution of differential equations - *Let's go back to our initial problem: we want to use LTs to solve differential equations.*
We will just apply the property of derivation to the nth order of a function.

Consider this differential equation:

$$\begin{cases} \frac{d^2y}{dt} + 3\,\frac{dy}{dt} + 2y(t) = (1+3t)u(t) \\ y(0^-) = 1 \\ \frac{dy}{dt}|_{t=0^-} = 0 \end{cases}$$

If $u(t) = 1(t)$, let's apply the property of LT for the derivatives:

$$\left\{s^2\,Y(s) - s\,y(0^-) - \dot{y}(0^-)\right\} + \{3\,[s\,Y(s) - y(0^-)]\} + 2\,Y(s) = \tfrac{1}{s} + \tfrac{3}{s^2}$$
$$\left\{s^2\,Y(s) - s\right\} + \{3\,[s\,Y(s) - 1]\} + 2\,Y(s) = \tfrac{1}{s} + \tfrac{3}{s^2}$$
$$(s^2 + 3\,s + 2)Y(s) - s - 3 = \tfrac{1}{s} + \tfrac{3}{s^2}$$

This is an algebraic equation in s we can easily solve for $Y(s)$:

$$Y(s) = \frac{s+3}{s^2+3\,s+2} + \frac{1}{s^2+3\,s+2}\,(\frac{1}{s} + \frac{3}{s^2}) = Y_l(s) + Y_f(s)$$

The first term $Y_l(s)$ is the *free response*, the second term $Y_f(s)$ is the *Forced response*. Now we need to go back to the time domain, since we are looking for $y(t)$!

6 Inverse Laplace Transformation

Under certain assumptions, one can uniquely define the Inverse Laplace Transform (ILT) of $F(s) = \mathcal{L}[f(t)]$ as $f(t) = \mathcal{L}^{-1}[F(s)]$; the operator $\mathcal{L}^{-1}$ is linear. The general formula is:

$$f(t) = \frac{1}{2\pi j} \int_{\bar{\sigma}-j\infty}^{\bar{\sigma}+j\infty} F(s)\, e^{st} ds \tag{3}$$

where $\bar{\sigma} > \sigma_0$ delimits the ROC and the integration is along the vertical line $\sigma = \bar{\sigma}$.

Integrals of complex variables

Let's recall some useful definitions from complex analysis. A function $f(t)$ which is $\mathcal{C}^\infty$ (or analytic) on the entire complex plane except for some isolated singularity a (i.e. the function is not continuous on a) can be approximated in *Laurent series* in a:

$$f(s) = \sum_{n=-\infty}^{+\infty} a_n\,(s-a)^n$$

The discontinuity point (called also singularity) a can be classified as:

- Removable: there are no negative powers in the Laurent series, therefore $\lim_{s\longrightarrow a} f(s) = a_0$. Thus we can define $f(a) = a_0$
- Pole of order m: there is a finite number m of negative powers in the Laurent series. Therefore a is a removable singularity for the function $f_1(s) = (s-a)^m f(s)$ and $\lim_{s\longrightarrow a} f(s) = \infty$.
- Essential singularity: there are infinite negative powers in the Laurent series.

If $f(s)$ is analytic on an area of the complex plane delimited by a closed path Γ, except for a finite number r of isolate singularities a_k in the interior of Γ, Cauchy's theorem tells us that:

$$\int_\Gamma f(s)ds = 2\pi j \sum_{k=1}^{r} Res(f, a_k)$$

Where $Res(f, a_k)$ is called integral residue of $f(s)$ on a_k.
It can be proved that the valued of $Res(f, a_k)$ is the coefficient $a_{k,-1}$ of the Laurent series expansion of $f(s)$ around a_k. **There exist easy formulas to compute the residuals, and we will see them later.**

$$Res(f, a_k) = a_{k,-1}$$

A Laplace Transform $F(s)$ is analytic in its domain of convergence, therefore all its singularities are to the left of the ROC line $\sigma = \sigma_0$.
Consider:

$$\frac{1}{2\pi j} \int_C F(s)e^{st} ds \tag{4}$$

Where C is called Bronwich integral path and is the solid bold line path shown in Figure 9.
If Γ is the semi-circle part of C, having radius R, we can write our ILT integral introduced at

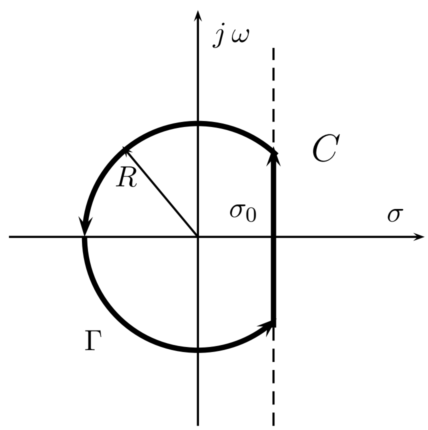


Figure 9: Inverse transformation integral path

equation (3) as:

$$\begin{aligned} f(t) &= \frac{1}{2\pi\, j}\int_{\bar{\sigma}-j\infty}^{\bar{\sigma}+j\infty} F(s)\, e^{st} ds = \\ &= \frac{1}{2\pi\, j}\left[\lim_{R\longrightarrow\infty}\int_C F(s)\, e^{st} ds - \lim_{R\longrightarrow\infty}\int_\Gamma F(s)\, e^{st} ds\right] \end{aligned}$$

If the term $\lim_{R\longrightarrow\infty}\int_\Gamma F(s)\, e^{st} ds$ was negligible, we could easily compute (3) using the residual formulas!
A sufficient condition yielding $\lim_{R\longrightarrow\infty}\int_\Gamma F(s)\, e^{st} ds = 0$ is that $F(s)$ is a rational function of polynomials in s, $F(s) = \frac{P(s)}{Q(s)}$, where $deg(Q) > deg(P)$ ($F(s)$ strictly proper, or causal!).
So we can apply Cauchy's theorem; the factor $2\pi j$ is eliminated. Summarizing, if $F(s)$ is analytic except for a finite number of singularities and strictly proper, then we can find the ILT as:

$$f(t) = \sum_{k=1}^{r} Res(F(s)e^{st}, a_k)$$

Note that if $R \to \infty$, then C includes all the singular points of $F(s)$.

6.1 Inverse Laplace Transform of polynomial rational functions

This is the most common type of transfer function, and we can mechanically find the ILT. Consider:

$$F(s) = \frac{N(s)}{D(s)}, \quad deg(D) = n,\, deg(N) = m, \quad m < n \tag{5}$$

This can be rewritten as:

$$F(s) = \frac{N(s)}{D(s)} = K\,\frac{(s-z_1)^{m_1}(s-z_2)^{m_2}\cdots(s-z_r)^{m_r}}{(s-p_1)^{n_1}(s-p_2)^{n_2}\cdots(s-p_q)^{n_q}}$$

con $m_1 + m_2 + ... + m_r = m$, ed $n_1 + n_2 + ... + n_q = n$.

- $z_1, z_2, ..., z_m \in \mathbb{C}$ are the distinct **zeros** of $F(s)$, each having multiplicity m_j;

- $p_1, p_2, ..., p_m \in \mathbb{C}$ are the distinct **poles** of $F(s)$, each having multiplicity n_i.

We can always decompose our rational function (5) in simple fractions:

$$\begin{aligned} F(s) = \quad & \frac{C_{1,1}}{(s-p_1)} + \quad ... \quad + \frac{C_{1,n_1}}{(s-p_1)^{n_1}} + \\ & + \frac{C_{2,1}}{(s-p_2)} + \quad ... \quad + \frac{C_{2,n_2}}{(s-p_2)^{n_2}} + \\ & + \frac{C_{q,1}}{(s-p_q)} + \quad ... \quad + \frac{C_{q,n_q}}{(s-p_q)^{n_q}} \end{aligned} \tag{6}$$

For the linearity property, which holds evidently also for the ILT:

$$\mathcal{L}^{-1}[F(s)] = \sum_{i=1}^{q} \sum_{j=1}^{n_i} \mathcal{L}^{-1}[\frac{C_{i.j}}{(s-p_i)^j}]$$

The ILT of each simple fraction has the form:

$$\mathcal{L}^{-1}[\frac{C_{i.j}}{(s-p_i)^j}] = \frac{C_{i,j}}{(j-1)!} \, t^{(j-1)} \, e^{p_i t} \, 1(t)$$

In general, $C_{i,j}$, $p_i \in \mathbb{C}$. In summary:

$$\mathcal{L}^{-1}[F(s)] = \sum_{i=1}^{q} \sum_{j=1}^{n_i} \frac{C_{i,j}}{(j-1)!} \, t^{(j-1)} \, e^{p_i t} \, 1(t)$$

Now: How do we calculate the coefficients $C_{i,j}$?

- **Method 1:** determine $C_{i,j}$ using the identity principle of polynomials. In practice, pick (6), multiply out everything and equate the coefficients of each power of s at the numerator to $N(s)$.

- **Method 2:** compute the residuals of $F(s)$: we are dealing with a strictly proper polynomial rational functions, with isolated singularities - the poles.
  It can be shown that $C_{i,j}$ are the coefficients a_{-1} of the Laurent series of $F(s)$ around the ith pole (note, of $F(s)$ and not of $F(s)e^{st}$).
  Residuals formula:

  $$C_{i,j} = \frac{1}{(n_i - j)!} \lim_{s \longrightarrow p_i} \left\{ \frac{d^{(n_i - j)}}{d\, s^{(n_i - j)}} F(s) \, (s - p_i)^{n_i} \right\} \tag{7}$$

  For simple poles (multiplicity one), this simplifies to:

  $$C_i = \frac{N(p_i)}{D'(p_i)} \tag{8}$$

  Where $D'(p_i)$ is $D(s)$ without the term $(s - p_i)$, evaluated at p_i.

Note: now we have decomposed our $F(s)$ into **parts that we know how to inverse transform**

Proof of formula (7). Consider $i = 1$:

$$F(s) = \sum_{j=1}^{n_1} \frac{C_{i,j}}{(s-p_1)^j} + K(s)$$

where $K(s)$ represents the remaining terms of the expansion. Taking the limit $s \longrightarrow p_1$:

$$\lim_{s \longrightarrow p_1} \left\{ (s-p_1)^{n_1} \sum_{j=1}^{n_1} \frac{C_{i,j}}{(s-p_1)^j} + (s-p_1)^{n_1} K(s) \right\} = C_{1,n_1}$$

since p_1 is not a pole of $K(s)$ and the second term goes to zero.
Consider then:

$$\frac{d}{ds}[(s-p_1)^{n_1} F(s)] = (n_1-1)(s-p_1)^{n_1-2} C_{1,1} + ... + C_{1,n_1-1}$$

taking again the limit one obtains C_{1,n_1-1} and further on for the remaining terms.

Examples

Let's go back to example 5.3), expanding in partial fractions:

$$Y(s) = Y_l + Y_f = \frac{s+3}{s^2+3\,s+2} + \frac{1}{s^2+3\,s+2}\left(\frac{1}{s} + \frac{3}{s^2}\right)$$

- Free response Y_l.
  Poles: $p_1 = -1$, $p_2 = -2$.

$$Y_l = \frac{C_1}{(s+1)} + \frac{C_2}{(s+2)}$$

  Apply **method 1**:

$$Y_l = \frac{C_1(s+2) + C_2(s+1)}{(s^2+3s+2)}$$

$$Y_l = \frac{s(C_1+C_2) + 2\,C_1 + C_2}{(s^2+3s+2)} = \frac{s+3}{s^2+3\,s+2}$$

  Equating the numerators:

$$\begin{cases} C_1 + C_2 = 1 \\ 2\,C_1 + C_2 = 3 \end{cases}$$

  We get $C_1 = 2$, $C_2 = -1$. Now inverse transform all the parts of Y_l.

$$Y_l = \frac{2}{(s+1)} - \frac{1}{(s+2)} \Longrightarrow \mathcal{L}^{-1}[Y_l] \Longrightarrow y_l(t) = 2\,e^{-t} - e^{-2t}$$

- Forced response Y_f.
  Poles: $p_1 = -1, p_2 = -2$ are simple poles, while $p_3 = 0$ has multiplicity 2. Partial fractions:

$$Y_f = \frac{C_1,1}{(s+1)} + \frac{C_2,1}{(s+2)} + \frac{C_{3,1}}{s^2} + \frac{C_{3,2}}{s}$$

  It's more convenient to use **method 2**.

Apply the residual formulas:

$$C_{1,1} = \frac{(s+3)}{(s^2)(s+2)}_{|s=-1} = 2$$

$$C_{2,1} = \frac{(s+3)}{(s^2)(s+1)}_{|s=-2} = -\frac{1}{4}$$

$$C_{3,1} = \frac{d}{ds}\frac{(s+3)}{(s+1)(s+2)}_{|s=0} = -\frac{7}{4}$$

$$C_{3,2} = \frac{(s+3)}{(s+1)(s+2)}_{|s=0} = \frac{3}{2}$$

Therefore we can inverse transform as:

$$y_f(t) = \frac{3}{2}t - \frac{7}{4} + 2\,e^{-t} - \frac{1}{4}e^{-2t}$$

The overall system response is the sum of $y_l(t)$ and $y_f(t)$.

7 The mass spring system revisited

Let's look again at the mass spring system in Figure 10. We know how to obtain the

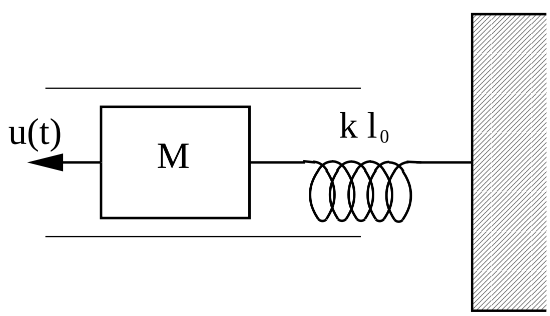


Figure 10: Spring mass system

differential equation that describes it:

$$M\frac{d^2(l-l_0)}{dt^2} + k(l-l_0) + b\frac{d(l-l_0)}{dt} = u(t)$$

Defining $y = (l - l_0)$ one gets:

$$M\frac{d^2y}{dt^2} + k\,y + b\frac{dy}{dt} = u(t)$$

Let's use the Laplace Transform:

$$M\left[s^2Y(s) - sy(0^-) - \dot{y}(0^-)\right] + kY(s) + b\left[sY(s) - y(0^-)\right] = U(s)$$

Solve for $Y(s)$:

$$Y(s)(Ms^2 + bs + k) = sMy(0^-) + M\dot{y}(0^-) + by(0^-) + U(s)$$

$$Y(s) = \frac{(sM+b)y(0^-) + M\dot{y}(0^-)}{Ms^2+bs+k} + \frac{U(s)}{Ms^2+bs+k}$$

Now we have the **free response** and the **forced response**. Note that

$$\frac{1}{Ms^2+bs+k}$$

is the LT of the impulse response of the system (we lose information about the initial conditions.) Now let's compare this with the state space description of the system, using variables x_1 (position) and x_2 (velocity):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now apply the LT to the matrix dynamical system:

$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x \end{cases}$$

$$\begin{aligned}
&\mathcal{L}[x] = \mathcal{L}[A\,x + B\,u] \\
&s\,X(s) - x(0^-) = A\,X(s) + B\,U(s) \\
&X(s)[s\,I - A] = x(0^-) + B\,U(s) \\
&X(s) = [s\,I - A]^{-1}x(0^-) + [s\,I - A]^{-1}B\,U(s) \\
&\implies \mathbf{Y(s) = C\,X(s) = C\,[s\,I - A]^{-1}x(0^-) + C\,[s\,I - A]^{-1}B\,U(s)} \implies \mathcal{L}^{-1}[\cdot] \implies \mathbf{y(t)}
\end{aligned}$$

Recall the inversion rule for a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This yields:

$$\begin{aligned}
[sI - A] &= \begin{bmatrix} s & -1 \\ \frac{k}{M} & s + \frac{b}{M} \end{bmatrix} \\
[sI - A]^{-1} &= \frac{M}{Ms^2+bs+k} \begin{bmatrix} s + \frac{b}{M} & 1 \\ -\frac{k}{M} & s \end{bmatrix}
\end{aligned}$$

Finally, multiplying by B, C we get $Y(s)$:

$$Y(s) = C[sI - A]^{-1}x(0^-) + C[sI - A]^{-1}BU(s)$$

Free response:

$$Y_l(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{M}{Ms^2+bs+k} \begin{bmatrix} s + \frac{b}{M} & 1 \\ -\frac{k}{M} & s \end{bmatrix} \begin{bmatrix} x_1(0^-) \\ x_2(0^-) \end{bmatrix} = \frac{M}{Ms^2+bs+k} \begin{bmatrix} s + \frac{b}{M} & 1 \end{bmatrix} \begin{bmatrix} x_1(0^-) \\ x_2(0^-) \end{bmatrix}$$

$$Y_l(s) = \frac{(sM+b)x_1(0^-) + Mx_2(0^-)}{Ms^2+bs+k}$$

Forced response:

$$Y_f(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{M}{Ms^2+bs+k} \begin{bmatrix} s+\frac{b}{M} & 1 \\ -\frac{k}{M} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} U(s)$$
$$= \frac{U(s)}{Ms^2+bs+k}$$

They are identical to what found just by applying the LT to the differential equation.
Note: we also found another way to compute e^{At}.

$$e^{At} = \mathcal{L}^{-1}[[sI-A]^{-1}]$$

Now let's look at the **step response** of the system, with zero initial conditions.

$$Y(s) = \frac{1}{s(Ms^2+bs+k)} = \frac{C_1}{s} + ...?...$$

The poles can be either real or complex:

$$p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4\,k\,M}}{2\,M}$$

1. Real and distinct poles: $b^2 > 4\,k\,M$.
   If $b = 1$, $M = 1/4$, $k = 5/36$ we get:

   $$y(t) = [\frac{36}{5} - 9e^{-\frac{1}{6}t} + \frac{9}{5}e^{-56t}]\,1(t)$$

2. Coincident poles: $b^2 = 4\,k\,M$.
   If $b = 1$, $M = 1/4$, $k = 1/4$ we get:

   $$y(t) = [4 - 4e^{-\frac{1}{2}t} - 2te^{-\frac{1}{2}t}]\,1(t)$$

3. **Most interesting case:** Complex conjugate poles: $b^2 < 4\,k\,M$.
   For instance:

   $$Y(s) = \frac{1}{s(s^2+s+1)}$$

   has $p_1 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$, $p_2 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$.

   Comparing with the " classical notation":

   $$Y(s) = \frac{1}{s(1 + \frac{2\xi}{\omega_n}s + \frac{s^2}{\omega_n^2})}$$

   the poles are placed as in Figure 11. If the damping ξ is low, the poles move towards the

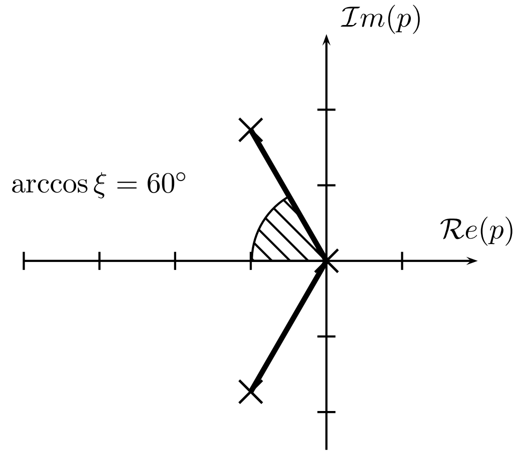


Figure 11: Complex conjugate poles

imaginary axis; for $\xi = 0$ we will have permanent oscillations. Finding the ILT:

$$Y(s) = \frac{C_1}{s} + \frac{C_2}{s + (\frac{1}{2} + j\frac{\sqrt{3}}{2})} + \frac{C_3}{s + (\frac{1}{2} - j\frac{\sqrt{3}}{2})}$$

Let's expand with **method 1**:

$$Y(s) = \frac{As + B}{s^2 + s + 1} + \frac{C}{s} = \frac{1}{s(s^2 + s + 1)}$$

$$Y(s) = \frac{Cs^2 + Cs + C + As^2 + Bs}{s(s^2 + s + 1)} = \frac{1}{s(s^2 + s + 1)}$$

$$\Longrightarrow \begin{cases} C = 1 \\ A = -1 \\ B = -1 \end{cases}$$

The response is:

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s+1}{(s^2 + s + 1)} = \frac{1}{s} - \frac{s+1}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \\ &= \frac{1}{s} - \left\{ \frac{s + \frac{1}{2}}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} + \frac{\boldsymbol{\sqrt{3}}}{\mathbf{3}} \frac{\boldsymbol{\frac{1}{2}\sqrt{3}}}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \right\} \\ &\Longrightarrow \mathcal{L}^{-1}[\cdot] \Longrightarrow \\ y(t) &= 1(t) - e^{-\frac{1}{2}t} \cos(\frac{\sqrt{3}}{2}t) - \frac{\sqrt{3}}{3} e^{-\frac{1}{2}t} \sin(\frac{\sqrt{3}}{2}t) \\ y(t) &= 1(t) - e^{-\frac{t}{2}} [\cos(\frac{\sqrt{3}}{2}t) + \frac{\sqrt{3}}{3} \sin(\frac{\sqrt{3}}{2}t)] \end{aligned}$$

If the sin coefficient is the tangent of $\pi/6$

$$y(t) = 1(t) - e^{-\frac{t}{2}}[\cos(\frac{\sqrt{3}}{2}t) + \frac{\sin\frac{\pi}{\mathbf{6}}}{\cos\frac{\pi}{\mathbf{6}}}\sin(\frac{\sqrt{3}}{2}t)]$$

$$\Rightarrow\ y(t) = 1(t) - e^{-\frac{t}{2}}\frac{1}{\cos\frac{\pi}{6}}[\cos(\frac{\sqrt{3}}{2}t - \frac{\pi}{6})]$$