CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b

John Doyle | **Problem Set #3** | Issued: 20 Feb
Winter 2014 | | Due: 7 Mar, 5 pm

1. **To infinity and beyond!**

   A signal x is said to be in $\mathcal{L}_2[0,\infty]$ if

   $$\|x\|_2^2 := \int_0^\infty x(t)^\top x(t)dt < \infty. \tag{1}$$

   As explained in class, the $\mathcal{H}_\infty$ norm of a stable system

   $$G := \begin{cases} \dot{x} = Ax + Bw \\ y = Cx \end{cases} \tag{2}$$

   is the induced norm from $\mathcal{L}_2[0,\infty] \to \mathcal{L}_2[0,\infty]$, i.e.

   $$\|G\|_\infty = \sup_{\|w\|_2=1} \|Gw\|_2 \tag{3}$$

   (a) Show that G satisfies $\|G\|_\infty < \gamma$ if and only if $\|y\|_2 < \gamma\|w\|_2$ for all $w \in \mathcal{L}_2[0,\infty]$.

   (b) Show that a sufficient condition for $\|G\|_\infty < \gamma$ is that there exists a matrix $P \succ 0$ such that

   $$\begin{bmatrix} A^\top P + PA + C^\top C & PB \\ B^\top P & -\gamma^2 \end{bmatrix} \prec 0 \tag{4}$$

   [*Note: this is also a necessary condition, and is the famous Kalman-Yakubovich-Popov (KYP) Lemma. The necessary direction is much harder to prove, and will be covered in CDS 212*]

   (c) Derive the dual to

   $$\begin{aligned} \min_{\gamma^2,\, P\succ 0} \quad & \gamma^2 \\ \text{s.t.} \quad & \begin{bmatrix} A^\top P + PA + C^\top C & PB \\ B^\top P & -\gamma^2 \end{bmatrix} \prec 0 \end{aligned} \tag{5}$$

   (d) Show that the dual problem provides a lower bound on $\|G\|_\infty$ using the following matrices (how to use them should be clear if you've derived the correct dual formulation):

   $$Z_{11} = \int_0^\infty x(t)x(t)^\top dt,\ Z_{12} = \int_0^\infty x(t)w(t)^\top dt,\ Z_{22} = \int_0^\infty w(t)w(t)^\top dt \tag{6}$$

   where $\|w\|_2 = 1$.

2. **Hanging in the balance** Implement a Kalman filter for the pendulum in the down position for $l_0 = 1$, $l_0 = .75$ and $l_0 = .5$, and plot the true and estimated trajectories on the same plot, as well as another plot comparing the MS estimation errors for the varying lengths of l_0.

Start your system from all zero initial conditions, except for $\theta_0 = \frac{\pi}{16}$, and have your system be driven by i.i.d. process noise $w_t \sim \mathcal{N}(0, .01I)$, and your measurements corrupted by i.i.d. sensor noise $v_t \sim \mathcal{N}(0, .01)$.

[*Note: the discrete time results from class can be converted to continuous time in a manner analogous to what was done for the LQR problem*].

3. **Balancing act** Modify the Kalman filter for the up position, and combine it with the LQR controller from Homework 2 in order to tackle the inverted pendulum stabilization problem that was assigned in Homework 1. Hand in a sample trajectory (subject to the same noise parameters as in problem 2), and a Nyquist plot.

   If you're feeling ambitious, see how your output-feedback controller performs in trajectory tracking as well as compared to the LQR solution that was implemented in Homework 2.