Finite-Horizon Optimal Control and Stabilization of Time-Scalable Systems

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Abstract

In this paper, we consider the optimal control of time-scalable systems. The time-scaling property is shown to convert the PDE associated with the Hamilton-Jacobi-Bellman (HJB) equation to a purely spatial PDE. Solution of this PDE yields the value function at a fixed time, and that solution can be scaled to find the value function at any point in time. Furthermore, in certain cases the unscaled control law stabilizes the system, and the unscaled value function acts as a Lyapunov function for that system. For the example of the nonholonomic integrator, this PDE is solved, and the resulting optimal trajectories coincide with the known solution to that problem.

1 Introduction

In addressing optimal control problems, one frequently looks for structures in the dynamical system being controlled which influence the structure of optimal trajectories and which simplify the solution of the problem. In this paper, we will discuss one such structure, namely the optimal control of dynamical systems which are time-scalable, meaning the equations of motion remain the same when time is reparameterized.

Examples of time scalable dynamic systems include driftless systems and mechanical systems without potential energy. Application areas include locomotion of systems with nonholonomic constraints,, control of underwater vehicles, and satellite reorientation [1, 4, 2]. These systems are not linearly controllable (when underactuated) and analysis of these systems has relied heavily on concepts and tools from differential geometry.

Optimal control of these classes of systems has been studied previously [5, 6], primarily in the context of simplifying the Euler-Lagrange equations using symmetry reduction. In contrast, we present some preliminary results regarding the structure of the Hamilton-Jacobi-Bellman (HJB) equation using the notion of time scalability. Specifically, we show that if the cost function is compatible with the time scaling, one can deduce *a priori* the time-dependence of the value function associated with the HJB PDE. This results in conversion of that PDE, which is normally varies in space and time, into a purely spatial PDE. The solution of this PDE, combined with the known time dependency, yields the control law which generates the optimal finite time trajectories. If the time variation of the control law is omitted, the same value function can produce a stabilizing (though not optimal) control law, with the value function playing the role of Lyapunov function.

The paper is structured as follows. In Section 2, we define time scalability and derive some properties of time-scalable dynamic systems. In Section 3, we discuss the application of time scalability to the optimal control problem. In Section 4, we solve the PDE for a simple example and show via simulation how it produces both finite-time and infinite-time stabilizing trajectories. Finally, in the conclusion we discuss future directions for research in this framework.

2 Time Scaling

2.1 Definitions

Let $x \in \mathbb{R}^p$ $u \in \mathbb{R}^s$. Let an nth order control system be a dynamical system of the form $x^{(n)} = f(x, \dot{x}, \ldots, x^{(n-1)}, u)$, where the superscript in parentheses indicates the order of differentiation. Let $\bar{x} \in \mathbb{R}^{pn}$ be a point of the form $\bar{x} = (x, \dot{x}, \ldots, x^{(n-1)})$.

Definition 1 An nth order control system is said to be *affinely time scalable* if, for any affine scaling of time $\tau = \alpha t + \beta$, there exists a scaling of u such that the scaled equations of motion are identical to the original equations.

Remark Let $K : \mathbb{R}^{pn} \times \mathbb{R}^s \times \mathbb{R} \to \mathbb{R}^{pn} \times \mathbb{R}^s \times \mathbb{R}$ denote the time scaling operator associated with a given time scaling. That is, $K(\bar{x}, u, t)$ maps each of the operands to its time-scaled counterpart. Because K acts on each operand indpendently (as well as acting on each $x^{(i)}$ independently, we will use, for example, $K(u)$ to denote the image of u under the time scaling.

We can easily write how K acts on the various operands by applying the chain rule. We see that

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} \tag{1}$$

$$= \alpha\frac{dx}{d\tau} \tag{2}$$

and thus we see that K maps $\dot{x}$ to $\dot{x}/\alpha$. More generally, we can see that $K(x^{(j)}) = x^{(j)}/\alpha^j$. Of course, $K(x) = x$. To determine if a control system is time-scalable, one simply makes this substitution and then sees if u can be compatibly scaled such that α can be factored out of the equation.

Example 1 First-order driftless systems, meaning systems of the form $\dot{x} = f(x)u$, are time scalable. If we scale time and substitute, we arrive at the equation

$$\frac{dx}{d\tau} = f(x)\frac{u}{\alpha} \tag{3}$$

If we define $K(u) = u/\alpha$, and substitute, we recover the original equation. More generally, nth-order driftless systems, meaning systems of the form $x^{(n)} = f(x)u$, are also time scalable, with the substitution $K(u) = u/\alpha^n$.

Example 2 Mechanical systems, including those with nonholonomic constraints, are time-scalable when the control vector fields are functions of the configuration space and the Lagrangian is purely kinetic energy. In coordinates, these systems take the form

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = g^{ij}(q)u_j \tag{4}$$

where Γ are the Christoffel symbols associated with the connection which defines the system. See [1] for more information on nonholonomic mechanical control systems. When $u = 0$, the time-scalability corresponds with the well-known time scalabilty of geodesics [3]. When u is nonzero, it can still be scaled by the procedure noted above.

Definition 2 A point $\bar{x}$ which exhibits the property $K(\bar{x}) = \bar{x}$ is said to be *scale invariant.* A scaling K for which $K(T) = T$, where $T \in \mathbb{R}$, is said to be *T-invariant.*

Remark The scale invariant points are clearly those for which all derivatives are zero. For a first-order control system, these consist of the entire space. A time scaling is T-invariant if $\beta = (1-\alpha)T$. We see that the function

$$r(t) = \frac{1}{(T-t)^n} \tag{5}$$

is compatible with any T-invariant time scaling, and the scale factor is $f(\alpha) = \alpha^{-n}$.

We now introduce a class of functions whose behaviour under time scaling we wish to investigate:

Definition 3 A function $L(x, \dot{x}, \ldots, x^{\{n-1\}}, u, t)$ is said to be *compatible* with a time-scaling operation K if the following holds:

$$L(K(x, \dot{x}, \ldots, x^{\{n-1\}}, u, t)) = f(\alpha)L(x, \dot{x}, \ldots, x^{\{n-1\}}, u, t) \tag{6}$$

Remark 1 An obvious property of $f(\alpha)$ is that $f(1) = 1$. Also, it is easy to see that $f(\alpha^{-1}) = f(\alpha)^{-1}$.

We see that the class of linearly time-scalable systems is fairly rich, and includes systems for which significant work has been done in the areas of controllability and trajectory generation. To our knowledge, however, the time scaling property of these systems has never explicitly been exploited. In the development which follows, we discuss the implications time scalability has for the optimal control problem.

2.2 Properties of Time-Scalable Systems

The following proposition follows immediately:

Proposition 1 *Given a time-scalable control system, initial conditions $\bar{x}_i$ at initial time $t = t_i$, and a control time history $u(t)$ defined on the interval $t \in [t_i, t_f]$, and a resulting trajectory $\bar{x}(t)$ defined on the interval, the time scaled control $K(u)$ defined on the interval $K([t_i, t_f])$, when applied to the time-scaled initial condition $K(\bar{x}_i)$ at time $K(t_i)$, will produce the time-scaled trajectory $K(\bar{x}(K(t)))$.*

Remark In other words, the time scaling of an integral curve yields another integral curve generated by the time-scaled control history. In particular, if the initial and final points are scale invariant, then a control time history which connects the two points in a given time interval can be scaled to connect those two points in any interval of time.

We now apply Proposition 1 to derive a property of integrals of compatible functions.

Proposition 2 *Given a time-scalable control system, a time scaling K for which $\alpha \geq 0$, a function $L(\bar{x}, u)$ which is compatible with K and for which $f(\alpha) \geq 0$, and a control history $u^*(t), t \in [t_i, t_f]$ which drives the system between $m_i \in M$ at time $t = t_i$ and $m_f \in M$ at time $t = t_f$ and minimizes the quantity*

$$J = \int_{t_i}^{t_f} L(\bar{x}, u) dt \tag{7}$$

Then the time-scaled control history $K(u^*(K(t)))$ minimizes J bewteen the time-scaled endpoints.

Proof To see the correspondence of J with its scaled counterpart, let us evaluate

$$\tilde{J} = \int_{K(t_i)}^{K(t_f)} L(\bar{x}, v)\, d\tau \tag{8}$$

with endpoints $K(m_i)$ and $K(m_f)$. We can evaluate the integral in the unscaled interval by changing coordinates through K^{-1}. Letting $t = K^{-1}\tau$, we apply the formula for change of coordinates of an integral to arrive at

$$\tilde{J} = \int_{t_i}^{t_f} L(K(\bar{x}, v))\alpha \, dt \tag{9}$$

where the extra α term is the Jacobian of the coordinate transformation. Replacing v with $K(u)$ and applying the definition of compatibility yields

$$\tilde{J} = \int_{t_i}^{t_f} \alpha f(\alpha) L(\bar{x}, u) \, dt \tag{10}$$

or

$$\tilde{J} = \alpha f(\alpha) J \tag{11}$$

which is clearly minimized by $u = u^*$, or $v = K(u^*)$, since the quantity $\alpha f(\alpha)$ is assumed positive.

Proposition 2 guarantees that $K(u^*)$ does in fact drive the system to the desired endpoint, and hence is an admissible control history. Furthermore, it guarantees that every control history which is admissible for the scaled function is also admissible for the original function. Hence no other control history could exist which yields a lower J than $v = K(u^*)$.

We now consider the properties of the integral function J for a special case. Suppose the final point m_f of the previous proposition is scale invariant, and let us fix the final time t_f. The function J can be thought of as a function of the initial point in M and the initial time. It is not an explicit function of u, since u is chosen according to the initial position and time. From the proof above, the following proposition holds:

Proposition 3 *Given the above conditions, $J(\bar{x}, t_i)$ is compatible with the set of all t_f-invariant scalings. Specifically,*

$$J(K(\bar{x}, t_i)) = \alpha f(\alpha) J(\bar{x}, t_i) \tag{12}$$

Finally, we are able to use compatibility to prove the following about the partial derivative of compatible functions:

Proposition 4 *Let $L(\bar{x}, u, t)$ be a function which is compatible with the set of all T-invariant time scalings. For simplicity, assume $K(u) = u/\alpha^n$. Then, for all $t_0 \neq T$,*

$$\left.\frac{\partial L}{\partial t}\right|_{t=t_0} = \frac{1}{t_0 - T}\left(\left.\frac{\partial f}{\partial \alpha}\right|_{\alpha=1} L + \sum_{i=1}^{n-1} i \frac{\partial L}{\partial x^{(i)}} x^{(i)} + n \frac{\partial L}{\partial u} u \right) \tag{13}$$

Proof Recall the definition of the partial derivative:

$$\left.\frac{\partial L}{\partial t}\right|_{t=t_0} = \lim_{\Delta t \to 0} \frac{L(x, \dot{x}, \ldots, x^{(n-1)}, u, t_0 + \Delta t) - L(x, \dot{x}, \ldots, x^{(n-1)}, u, t_0)}{\Delta t} \tag{14}$$

We can map the first term on the right hand side using the time scaling operator. If we set

$$\alpha = \frac{t_0 - T}{t_0 + \Delta t - T} \tag{15}$$

$$\beta = \frac{T \Delta t}{t_0 + \Delta t - T} \tag{16}$$

we recover the unique T-invariant time-scaling for which $K(t_0 + \Delta t) = t_0$. Thus,

$$L(x, \dot{x}, \ldots, x^{(n-1)}, u, t_0 + \Delta t) = \frac{1}{f(\alpha)} L\left(x, \frac{\dot{x}}{\alpha}, \ldots, \frac{x^{(n-1)}}{\alpha^{n-1}}, \frac{u}{\alpha^n},\ t_0\right) \tag{17}$$

Now $\alpha^{-1} = 1 + \Delta t/(t_0 - T)$, and $\alpha^{-n} = 1 + n\Delta t/(t_0 - T) + \mathcal{O}(\Delta t^2)$. Using this, we can Taylor expand the right hand side to first order, to arrive at

$$L(x, \dot{x}, \ldots, x^{(n-1)}, u, t_0 + \Delta t) = \frac{1}{f(\alpha)} \left[L + \left(\sum_{i=1}^{n-1} i \frac{\partial L}{\partial x^{(i)}} x^{(i)} + n \frac{\partial L}{\partial u} u \right) \frac{\Delta t}{t_0 - T} \right] + \mathcal{O}(\Delta t^2) \tag{18}$$

where the right-hand side is evaluated at $(x, \dot{x}, \ldots, , x^{(n-1)}, u, t_0)$. If we recall that at $\Delta t = 0$, we have $\alpha = 1$ and thus $f(\alpha)$ goes to 1 in the limit. Substituting into Equation (14) and taking the limit, we have

$$\left.\frac{\partial L}{\partial t}\right|_{t=t_0} = \frac{1}{t_0 - T} \left(\sum_{i=1}^{n-1} i \frac{\partial L}{\partial x^{(i)}} x^{(i)} + n \frac{\partial L}{\partial u} u \right) + \lim_{\Delta t \to 0} \frac{1 - f(\alpha)}{f(\alpha)\Delta t} L \tag{19}$$

The final term can be simplified by noting that $\alpha = 1 - \frac{\Delta t}{t_0 + \Delta t - T}$, Taylor expanding $f(\alpha)$ to first order, and taking the limit as before. This results in the following:

$$\left.\frac{\partial L}{\partial t}\right|_{t=t_0} = \frac{1}{t_0 - T} \left(\left.\frac{\partial f}{\partial \alpha}\right|_{\alpha=1} L + \sum_{i=1}^{n-1} i \frac{\partial L}{\partial x^{(i)}} x^{(i)} + n \frac{\partial L}{\partial u} u \right) \tag{20}$$

which proves our proposition.

The significance of this proposition is that it shows how the *temporal* partial derivative can be converted to a *spatial* quantity which includes time as a parameter.

3 Optimal Control of Time-Scalable Systems

3.1 Application of Time-Scalability to the HJB equation

We are now ready to consider the relationship between time scaling and the optimal control problem. Recall that the value function $V(\bar{x}, t)$ associated with the Hamilton-Jacobi-Bellman (HJB) equation is defined as the minimum cost necessary to drive a system from a point $\bar{x}$ at time t to a desired final condition at time (not necessarily finite) T. To apply the above theory to this problem, we consider a finite time optimal control problem with fixed final state which is scale invariant, and a cost function $L(\bar{x}, u, t)$ which is compatible with all time scalings. Given this, we see that the value function is given by

$$V(\bar{x}, t) = \int_t^T L(\bar{x}, u, \tau) d\tau \tag{21}$$

Applying Proposition 3, we see that $V(\bar{x}, t)$ is compatible with all T-invariant time scalings, and hence Proposition 4 applies as well.

The HJB equation for this optimal control problem is given by

$$\frac{\partial V}{\partial t} = -\min_u \left[L(\bar{x}, u, t) + \frac{\partial V}{\partial \bar{x}} f(\bar{x}, u) \right] \tag{22}$$

Proposition 5 allows us to replace the left-hand side with a term involving V and its spatial partial derivatives. Solving this spatial PDE yields the value function at one point in time, and the value function at any other time can be found using the time scaling. The presence of the $\frac{1}{t-T}$ term in Equation (13) indicates that as t approaches T, the value function will approach infinity at all points except the desired endpoint. This is logical - as time runs out, the cost associated with driving the system to the endpoint increases.

Ordinarily, the HJB equation is solved by defining $V(\bar{x}, T)$ using a terminal cost function and propagating that function backwards in time. In our case, the $\bar{x}(T)$ is fixed, so a terminal cost constraint is not meaningful. Nonetheless, the Principle of Optimality from which the HJB equation is derived is still applicable, so the use of that equation is still valid.

In the next sections, we consider the cases of first-order systems and affine connection control systems in greater detail. For the former, we shall show that solving the spatial PDE yields a stabilizing controller in addition to solving the finite-horizion optimal control problem.

3.2 First-Order Systems

Let us consider a system of the form

$$\dot{x} = g(x)u \tag{23}$$

where $g(x)$ is C^1, and a cost function

$$J = \frac{1}{2} \int_{t_0}^T u^T u \, dt \tag{24}$$

and suppose the constraint was that $x(T) = 0$. Evaluating the HJB equation for this system yields

$$V_t = \frac{1}{2} V_x^T g(x) g^T(x) V_x \tag{25}$$

and optimal control

$$u^*(x,t) = -g^T(x) V_x \tag{26}$$

where the subscripts on V indicate partial derivatives. We now turn to replacing V_t with its spatial counterpart. The cost function $u^T u$ is compatible with any time scaling, and the scaling function is $f(\alpha) = \alpha^{-2}$. The scaling for V, following Proposition 4, is $f(\alpha) = \alpha^{-1}$. Applying Proposition 4, we have

$$V_t = \frac{1}{T-t} V \tag{27}$$

Furthermore, applying the definition of time-scalability, we see that

$$V(x,t) = \frac{T-t_0}{T-t} V(x,t_0) \tag{28}$$

Letting $t_0 = 0$ and defining $\tilde{V}(x) = V(x,0)$, we can substitute into Equation (25) and derive the following spatial PDE:

$$\frac{1}{T}\tilde{V} = \frac{1}{2}\tilde{V}_x^T g(x) g^T(x) \tilde{V}_x \tag{29}$$

Solving this PDE yields the optimal finite-horizon control law by finding

$$\tilde{u}(x) = -g^T(x)\tilde{V}_x \tag{30}$$

and scaling $\tilde{u}(x)$ approriately in time to arrive at $u^*(x,t)$.

We now consider the possiblity of using $\tilde{u}(x)$ as a control law without time scaling, and asking whether that control law stablizes the system. When V is smooth, it is easy to show that it acts as a Lyapunov function for the control law $\tilde{u}(x)$.

$$\begin{aligned} \dot{\tilde{V}} &= \tilde{V}_x^T \dot{x} \\ &= \tilde{V}_x^T g(x) u^*(x) \\ &= -\tilde{V}_x^T g(x) g^T(x) \tilde{V}_x \\ &= -\frac{2}{T}\tilde{V} \end{aligned} \tag{31}$$

Since $\tilde{V}$ is a value function, $\dot{\tilde{V}}$ must be negative at all points $x \neq 0$. Thus, solving a single spatial PDE yields both optimal finite-horizon trajectories and an asymptotically stabilizing controller whose gain can be set. Note further that by varying the value of t_0, we can influence the rate of convergence of $\tilde{V}$, i.e. the gain of the controller. Whether or not the convergence of the system is exponential will depend on whether the value function is locally quadratic, a fact which will depend on the underlying system.

Finally, we note that Equation (24) is not the only compatible cost function. A function of the form

$$J = \int_{t_0}^{T} \frac{a}{(T-\tau)^2} x^T x + b u^T u \, d\tau \tag{32}$$

is also compatible. This cost function can be viewed as forcing $x(t)$ to approach zero sufficiently quickly to avoid the cost function becoming infinite.

3.3 Mechanical Control Systems

In this section, we consider systems of the form

$$\ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k = g^{ij}(q) u_j \tag{33}$$

as discussed in Section 1, with the cost function in Equation (24) In this case, the scaling function is $f(\alpha) = \alpha^{-4}$, and the scaling function for V is $\alpha f(\alpha) = \alpha^{-3}$. Thus, following Proposition 5, we have

$$V_t = \frac{1}{T-t}(3V - V_{\dot{q}} \dot{q}) \tag{34}$$

As in the first-order case, we can derive the following equation for V:

$$V(q, \dot{q}, t) = \frac{(T-t_0)^3}{(T-t)^3} V(q, \frac{T-t}{T-t_0} \dot{q}, t_0) \tag{35}$$

and it is easily verified that this form for V satisfies the equation for V_t. Again, we set $t_0 = 0$ and denote the spatial PDE as $\tilde{V}(x, \dot{x}) = V(q, \dot{q}, 0)$. We use x for the operands of $\tilde{V}$ since the operands of V and $\tilde{V}$ are not the same. Of course, they can be identified by

$$x = q, \; \dot{x} = \frac{T-t}{T} \dot{q} \tag{36}$$

a fact we shall use shortly, but we want to make clear the difference between partial differentiation with respect to the second operand of V, i.e. x_2, and with respect to q_2.

The HJB PDE for this system is

$$V_t = -V_q \dot{q} + V_{\dot{q}} \Gamma^i_{jk} \dot{q}^j \dot{q}^k + \frac{1}{2} V_{\dot{q}}^T g(q) g^T(q) V_{\dot{q}} \tag{37}$$

We take the partials of Equation (35) with respect to the relevant terms and substitute in Equations (35),(36), we arrive at

$$\frac{3}{T} \tilde{V} - \frac{1}{T} \tilde{V}_{\dot{x}} \dot{x} = -\tilde{V}_x \dot{x} + \tilde{V}_{\dot{x}} \Gamma^i_{jk} \dot{x}^j \dot{x}^k + \frac{1}{2} \tilde{V}_{\dot{x}}^T g(x) g^T(x) \tilde{V}_{\dot{x}} \tag{38}$$

If we attempt to use this value function as a Lyapunov function as done earlier, we arrive at the following equation:

$$\dot{\tilde{V}} = -\frac{3}{T} \tilde{V} - \frac{1}{2} \tilde{V}_{\dot{x}}^T g(x) g^T(x) \tilde{V}_{\dot{x}} + \frac{1}{T} \tilde{V}_{\dot{x}} \dot{x} \tag{39}$$

The above equation is not necessarily negative due to the last term. Given a solution $\tilde{V}$, this equation can be evaluated to determine if $\tilde{V}$ is in fact a Lyapunov function.

3.4 Example - The Nonholonomic Integrator

As an example of this theory, we consider a classic nonlinear optimal control problem, namely the first-order nonholonomic integrator. The system is defined by

$$\dot{x}_1 = u_1 \tag{40}$$

$$\dot{x}_2 = u_2 \tag{41}$$

$$\dot{x}_3 = x_1 u_2 - x_2 u_1 \tag{42}$$

with the cost function

$$J = \frac{1}{2}\int_{t_0}^{T} u_1^2 + u_2^2 \, dt \tag{43}$$

The system is driftless, linearly uncontrollable, and time-scalable. The optimal trajectories of this system are known to take the form

$$u_1 = k_1 \sin \mu t + k_2 \cos \mu t \tag{44}$$

$$u_2 = k_1 \cos \mu t - k_2 \sin \mu t \tag{45}$$

where k_1, k_2, μ are parameters chosen to meet the endpoint conditions. The HJB equation for this system is

$$\frac{\partial V}{\partial t} = \frac{1}{2}\frac{\partial V}{\partial x}^T \begin{pmatrix} 1 & 0 & -x_2 \\ 0 & 1 & x_1 \\ -x_2 & x_1 & x_1^2 + x_2^2 \end{pmatrix} \frac{\partial V}{\partial x} \tag{46}$$

Following the development in Section 3.1, we can rewrite this as

$$\tilde{V} = \frac{1}{2}\frac{\partial \tilde{V}}{\partial x}^T \begin{pmatrix} 1 & 0 & -x_2 \\ 0 & 1 & x_1 \\ -x_2 & x_1 & x_1^2 + x_2^2 \end{pmatrix} \frac{\partial \tilde{V}}{\partial x} \tag{47}$$

We now seek a functional form for V which is consistent with this equation. If we assume that $V = V(d, x_3)$ where $d = x_1^2 + x_2^2$ (a fact which can be deduced from the symmetry of the problem), we can substitute and simplify the equation to

$$\tilde{V} = 2d\frac{\partial \tilde{V}}{\partial d}^2 + \frac{d}{2}\frac{\partial \tilde{V}}{\partial x_3}^2 \tag{48}$$

If we further assume that V has the form

$$\tilde{V} = \frac{d}{2} f(r) \tag{49}$$

where $r = x_3/d$, we can reduce the equation to an ODE:

$$f = f^2 - 2rff' + (r^2 + \frac{1}{4})f'^2 \tag{50}$$

This equation can be factored to form two ODEs, namely

$$f' = \frac{2\left(2rf \pm \sqrt{4r^2f^2 - (4r^2+1)(f^2-f)}\right)}{4r^2+1} \tag{51}$$

To solve either ODE, we need an initial condition. Note that when $r = 0$, this is equivalent initial conditions in the $x_1 - x_2$ plane. Reviewing the equations of motion, we see that this case can be solved directly using constant $u_1 = x_1/T_r$,$u_2 = x_2/T_r$. This corresponds to $V(d,0) = \frac{d}{2T_r}$, meaning $f(0) = 1$. Note that for $f = 1, r \geq 0$, the ODE corresponding to the positive square root is a fixed point of the system, and for $r \leq 0$ the other one is. Thus, by solving the initial value problem for both positive and negative r using the ODE which does not give a trivial solution, we derive $f(r)$, and thus the value function as well.

At points away from the x_3 axis, we have the following control law:

$$\tilde{u}_1 = -x_1(f - f'r) + \frac{1}{2}x_2 f' \tag{52}$$

$$\tilde{u}_2 = -x_2(f - f'r) - \frac{1}{2}x_1 f' \tag{53}$$

Solving Equation (51) reveals that $\lim_{r\to\infty} f(r) = \infty$. However, $\lim_{r\to\infty} V(d,r)$ is finite, as well as the limit of $f'(r)$. This allows us to define V at points where $x_1,\ x_2 = 0$. Furthermore, $\lim_{r\to\infty} u_i$ is finite, but dependent on the path along which one takes the limit. It can be shown that along the x_3 axis a control law of the form

$$u_1 = \cos\theta\sqrt{2\tilde{V}(0,0,x_3)} \tag{54}$$

$$u_2 = \sin\theta\sqrt{2\tilde{V}(0,0,x_3)} \tag{55}$$

for arbitrary θ yields a control law which is continuous along trajectories. The nonuniqueness of the optimal control law at those points can be viewed as a consequence of the symmetry of the optimal control problem.

Figures 1 and 2 show optimal finite-time trajectories and stabilizing trajectories for two initial conditions. The stabilizing trajectories were generated using the value function at $t_0 = 9$ to give faster convergence. The initial conditions in Figure 1 are at a point where the value function is smooth. Those for Figure 2 are at a nonsmooth point. Both show the trajectory reaching zero at the appropriate time for the finite-horizon case, and decaying to zero in the infinite horizon case. For first-order systems such as this, the finite and infinite time trajectories are actually the same. Because the stabilizing controller does not have the $1/(t-T)$ term to increase the control effort, it takes infinite time to reach the origin.

The resulting trajectories are consistent with those given in Equations (44),(45). We also note that for certain initial conditions, multiple values of $k1, k2, \mu$ satisfy those equations and endpoint conditions, corresponding to locally minimizing solutions. The value function derived above returns the globally minimizing trajectory.

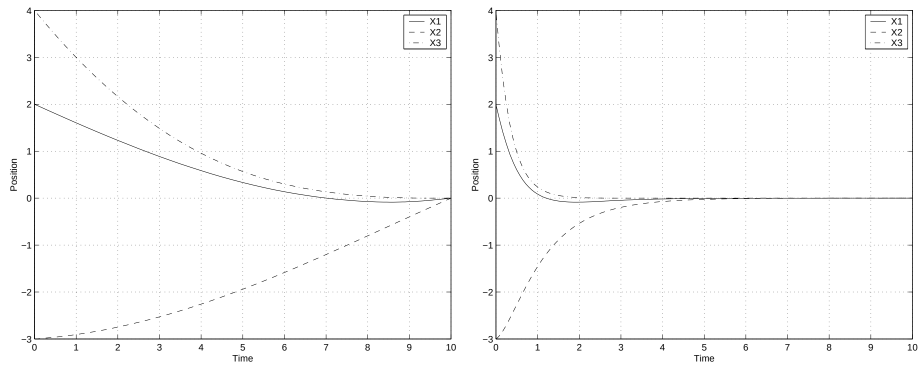

Figure 1: Finite-Time Optimal Trajectory, Stabilizing Trajectory, $x_0 = (2, -3, 4)$

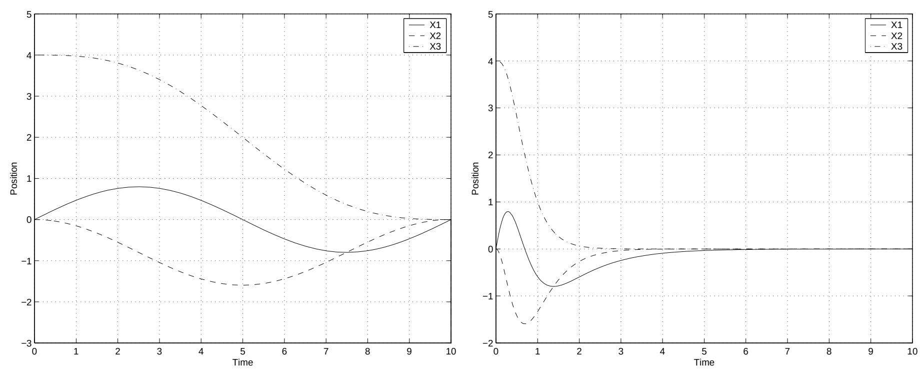

Figure 2: Finite-Time Optimal Trajectory, Stabilizing Trajectory, $x_0 = (0, 0, 4)$

4 Future Work

In the example shown above, we see that the solution to the spatial PDE is nonsmooth. This is unsurprising - the existence of a smooth solution for an underactuated driftless system would imply the existence of a continuous stabilizing feedback, which is known not to exist [**?**]. The discontinuity, however, is relatively benign, in that trajectories never cross it. It would be desirable to show that the discontinuities inherent in the solution do not prevent the control law from stabilizing the system. Such a proof would involve tools from nonsmooth analysis, and will be the subject of future research. Additionally, we seek to refine the conditions under which mechanical control systems can be stabilized. This will involve deriving conditions under which Equation (**??**) is known *a priori* to be negative. Issues of smoothness will have to be addressed in this setting as well.

In this paper, we showed how the HJB equation can be simplified through *a priori* knowledge of the solution structure corresponding to the structure of the underlying dynamical system and cost function. In practice, the resulting PDE will still not be solvable in closed form. Future research will focus on developing numerical methods of solution tailored to this framework. The literature regarding the geometric structure of the systems discussed in this paper is very rich. We expect to be able to exploit this structure in designing numerical methods.

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