CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 202

Problem Set #7

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Reading: Boothby, IV.1, IV.5–IV.8

1. [Guillemin and Pollack, page 160, #1, #2]

   (a) Suppose that $T \in \Lambda^p(V)$ and $v_1, \ldots, v_p \in V$ are linearly dependent. Prove that $T(v_1, \ldots, v_p) = 0$ for all $T \in \Lambda^p(V)$.

   (b) Suppose that $\omega_1, \ldots, \omega_p \in V^*$ are linearly dependent. Show that $\omega_1 \wedge \cdots \wedge \omega_p = 0$.

2. Let $T = 2e^1 \otimes e^1 - e^2 \otimes e^1 + 3e^1 \otimes e^2$ with $T \in \mathcal{T}_2(V)$ with $V = \mathbb{R}^3$. Let $\varphi \in L(\mathbb{R}^2, \mathbb{R}^2)$, $\psi \in L(\mathbb{R}^3, \mathbb{R}^2)$ be given by the matrices

$$\varphi = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

   and

$$\psi = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}.$$

   Compute trace(t), $\varphi^*(t)$, $\psi^*(t)$.

3. [Guillemin and Pollack, page 178, #1] Calculate the exterior derivatives of the following forms in $\mathbb{R}^3$:

   (a) $z^2 dx \wedge dy + (z^2 + 2y) dx \wedge dz$

   (b) $13x dx + y^2 dy + xyz dz$

   (c) $f dg$, where f and g are functions on $\mathbb{R}^3$

   (d) $(x + 2y^3)(dz \wedge dx + \frac{1}{2} dy \wedge dx)$

4. Use exterior derivatives to show that curl(grad f) = 0 and div(curl $\vec{F}$) = 0.

5. [Boothby, page 207, #3] If $\varphi_i \in \bigwedge^1(V)$, $i = 1, ..., k$, show that $\varphi_1 \wedge ... \wedge \varphi_k = det(\varphi_i(v_j))$, a $k \times k$ determinant.

6. [[Boothby, page 212, #2) Prove that the volume of the parallelepiped of $\mathbb{R}^3$ whose vertex is at the origin and whose sides (from this vertex) are the vectors $v_i = (x_i^1, x_i^2, x_i^3)$, $i = 1, 2, 3$ is in fact the determinant of the matrix (x_i^j).