CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 202
Problem Set #5

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Reading: Boothby, IV.7–IV.8 (Frobenius' Theorem)

1. Consider the following vector fields on $\mathbb{R}^3$:
$$X(x) = \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} \qquad Y(x) = \frac{\partial}{\partial x_1}.$$
Let $x_0 = (0,0,0)$. Show that $\phi_h^{-Y} \circ \phi_h^{-X} \circ \phi_h^{Y} \circ \phi_h^{X}(x_0) = h^2 \phi^{[X,Y]}(x_0)$.

2. Show that if Δ is a distribution of the form
$$\Delta = \operatorname{span}\{X_1, \ldots, X_d\}$$
and we have $[X_i, X_j] \in \Delta$ for all i, j then for *any* $X, Y \in \Delta$, $[X, Y] \in \Delta$. That is, to check involutivity of a distribution, we need only check that the pairwise brackets between basis elements lie in the distribution.

3. [Boothby, page 164, #4] Let $N \subset M$ be a submanifold and let $X, Y \in \mathcal{X}(M)$ be vector fields such that $X_p, Y_p \in T_pN$ for $p \in N$. Show that $[X, Y]_p \in T_pN$ for all $p \in N$.

4. [Boothby, page 164, #5] Let $F : M \to N$ be a smooth submersion of M onto N. Show that $F^{-1}(q)$ for all $q \in N$ are the leaves of a foliation on M.

5. Let $\operatorname{ad}_X Y := [X, Y]$ and define $\operatorname{ad}_X^k Y := [X, \operatorname{ad}_X^{k-1} Y]$ (i.e. $\operatorname{ad}_X^k Y$ contains k copies of X). Define a set of distributions on M, a smooth n-dimensional manifold, as
$$\Delta_i = \operatorname{span}\{Y, \operatorname{ad}_X Y, \ldots, \operatorname{ad}_X^i Y\}.$$
Show that if Δ_{n-1} is full rank (i.e. $\Delta_{n-1}(x) = T_xM$) and Δ_{n-2} is involutive, then Δ_i is involutive for all $i = 1, \ldots, n-1$.

6. Let $SO(3)$ be the set of 3×3 orthogonal matrices with determinant $+1$. The tangent space of $SO(3)$ at the identity is given by the set of skew-symmetric matrices of the form
$$\widehat{\omega} = (\omega)^\wedge = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$
   (a) Show that the tangent space $T_R SO(3)$ consists of matrices of the form $\widehat{\omega} R$ where $\widehat{\omega}$ is skew-symmetric.
   (b) Show that the flow of a vector field $g(R) = \widehat{\omega} R$ is given by $\phi_t(R) = \exp(\widehat{\omega} t) R$ where exp is the matrix exponential.
   (c) Show that the Lie bracket between two vector fields $g_1(R) = \widehat{\omega}_1 R$ and $g_2(R) = \widehat{\omega}_2 R$ is given by
   $$[g_1, g_2](R) = (\omega_1 \times \omega_2)^\wedge R,$$
   where $\times$ is the cross product in $\mathbb{R}^3$. (Hint: you may use the "fact" that $[X, Y](x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} (\phi_\epsilon^Y \phi_\epsilon^X - \phi_\epsilon^X \phi_\epsilon^Y)(x)$ if you need to).