CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 202
Problem Set #4

R. Murray
Winter 2004

Issued: 2 Feb 04
Due: 9 Feb 04

Reading:

Boothby, IV.1–IV.4

Problems:

1. [Boothby, page 118, #2, 3]
   Let M be a smooth manifold and define $C^\infty(M)$ as the set of all smooth real-valued functions on M.
   (a) Show that a C^∞-vector field X on M defines a derivation on $C^\infty(M)$ by $Xf(p) = X_p f$.
   (b) Show that the derivations of $C^\infty(M)$ are in natural one-to-one correspondence with $\mathcal{X}(\mathcal{M})$, the collection of all smooth vector fields on M.
2. [Boothby, page 119, #12]
   Show that any smooth vector field Y on $S^{n-1} \subset \mathbb{R}^n$ is the restriction of a smooth vector field X on $\mathbb{R}^n$.
3. [Boothby, page 126, #6]
   Show that $\phi_t(x, y)$ defined by
   $$\phi_t(x, y) = (xe^{2t}, ye^{-3t})$$
   defines a C^∞ flow on $M = \mathbb{R}^2$. Determine the infinitesimal generator of the flow and show that it is ϕ invariant.
4. [Boothby, page 134, #4]
   Let X and Y be vector fields on manifolds M and N, respectively, and $F : M \to N$ a smooth mapping. Show that X and Y are F-related if and only if the local flows ϕ and ψ generated by X and Y satisfy $F \circ \phi_t(p) = \psi_t \circ F(p)$ for all (t, p) for which both sides are defined.
5. In this problem we will explore the properties of the *Lie bracket* between two vector fields, $[X, Y] = XY - YX$.
   (a) Let X, Y be two smooth vector fields on M. Show that $XY : C^\infty(M) \to C^\infty(M)$ is not a vector field but that $XY - YX$ is.
   (b) Let $X, Y \in \mathcal{X}(\mathcal{M})$ and $f, g \in C^\infty(M)$. Prove that
   $$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$