CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 202
Problem Set #3

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Reading:

Boothby, Sections II.6–II.7 and III.4–III.5

Problems:

1. Recall from lecture how we defined a coordinate neighborhood (ϕ, U) for a manifold M:

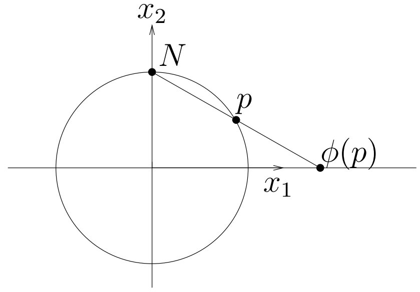
$$\begin{array}{c} U \subset M \subset \mathbb{R}^k \\ \phi \big\downarrow \\ \tilde{U} \subset \mathbb{R}^m \end{array} \qquad \begin{array}{l} M : m \text{ dimensional mfd} \\ U : \text{open subset of } M \\ \tilde{U} : \text{open subset of } \mathbb{R}^m \\ \phi : \text{diffeomorphism of } \tilde{U} \text{ onto } U \end{array}$$

If M is an embedded manifold in $\mathbb{R}^k$ $(k > m)$ then the definition of the diffeomorphism ϕ can be confusing at first glance. For instance, it would seem that ϕ should map an open set of $\mathbb{R}^k$ onto an open set of $\mathbb{R}^m$, but this is impossible if ϕ is a diffeomorphism. The proper interpretation is that ϕ *smoothly extends* to a map $\Phi : V \to \mathbb{R}^m$ (where $U = M \cap V$, $V \subset \mathbb{R}^k$ open), such that $\Phi|_U$ is a diffeomorphism onto $\tilde{U}$. We will explore these concepts in this exercise using $M = S^1$ and $N = 2$.

Let $S^1 \subset \mathbb{R}^2$ be the set given by

$$S^1 = \{x \in \mathbb{R}^2 : x^T x = 1\}.$$

(a) Construct the stereographic projection $\Phi : \mathbb{R}^2 \to \mathbb{R}$ which maps $S^1 - \{N\}$ to the real line as shown in the figure below. $\{N\}$ denotes the "north pole", i.e., $x = (0, 1)$.



(b) Let $U = S^1 - \{N\}$ and define $\phi = \Phi|_U$. Construct the map $\phi^{-1} : \mathbb{R} \to U \subset \mathbb{R}^2$ and show that it is a smooth mapping.

(c) Show that ϕ is a bijection by verifying that

   i. the map $\Phi \circ \phi^{-1} : \mathbb{R} \to \mathbb{R}$ is the identity;

   ii. $\phi^{-1} \circ \Phi$ is the identity *when restricted to U*.

(d) The tangent space at a point $p \in U$ is defined as

$$T_pS^1 := \mathrm{Im}\left(d\phi^{-1}_{\phi(p)}\right).$$

Show that this vector subspace of $\mathbb{R}^2$ can be identified with vectors tangent to S^1 at p. Hence, when talking about a tangent vector of $S^1 \subset \mathbb{R}^2$ it makes sense to only consider vectors in $\mathrm{Im}(d\phi^{-1})$.

(e) Show that $d\phi_p : T_pS^1 \to \mathbb{R}$ is an isomorphism by verifying that

i. $d\Phi_p \cdot d\phi^{-1}_{\phi(p)} = I \in \mathbb{R}$;

ii. $\left(d\phi^{-1}_{\phi(p)} \cdot d\Phi_p\right)\Big|_{T_pS^1} = \mathrm{id}$.

Use these calculations to interpret the symbol $d\phi_p$.

2. [Guillemin and Pollack, page 18, #2]
Suppose that P is an l-dimensional submanifold of M where the differentiable structure on P is inherited from M. That is, if o(ϕ, U) is a coordinate chart on M then $(\phi|_P, P \cap U)$ is a coordinate chart on P. Let $z \in P$. Show that there exists a local coordinate system $\{x_1, \ldots, x_k\}$ defined in a neighborhood U of z in M such that $P \cap U$ is defined by the equations $x_{l+1} = 0, \ldots, x_k = 0$.

3. [Guillemin and Pollack, page 18, #6]

(a) If f and g are immersions, show that $f \times g$ is.

(b) If f and g are immersions, show that $g \circ f$ is.

(c) If f is an immersion, show that its restriction to any submanifold of its domain is an immersion.

(d) When $\dim M = \dim N$, show that immersions $f : M \to N$ are the same as local diffeomorphisms.

4. [Guillemin and Pollack, page 18, #7]

(a) Show that $g : \mathbb{R}^1 \to S^1$ defined by $g(t) = (\cos 2\pi t, \sin 2\pi t)$ is a local diffeomorphism.

(b) Show that $G : \mathbb{R}^2 \to S^1 \times S^1$ defined by $G = g \times g$ is a local diffeomorphism.

(c) Show that if L is a line in $\mathbb{R}^2$ then the restriction $G : L \to S^1 \times S^1$ is an immersion and if L has irrational slope then G is one-to-one on L.

5. [Boothby, page 46, #5 and #6]
Prove the following two corollaries of the implicit function theorem:

(a) Let W be an open subset of $\mathbb{R}^n$ and $F : W \to \mathbb{R}^n$. If df_x is nonsingular at every point $x \in W$, then F is an open mapping of W. That is, it carries W and open subsets of W to open subsets of $\mathbb{R}^n$.

(b) A necessary and sufficient condition for the C^∞ map F to be diffeomorphism from W to $F(W)$ is that it be injective and df_x be nonsingular at every point $x \in W$.