CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

ACM/CDS 202
Problem Set #4

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Reading: Abraham, Marsden, and Ratiu (MTA), sections 3.3, 4.1, 4.2

Problems:

1. If $F(x_1, x_2, x_3) = 0$ is a submersion defining a 2-dimensional manifold in $\mathbb{R}^3$, under what conditions is $X = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v_3 \frac{\partial x}{\partial x_3}$ (evaluated at a point where $F(x_1, x_2, x_3) = 0$) a tangent vector to M?

2. MTA 4.1-5: convolution equation

3. [Boothby, page 126, #6]
Show that $\phi_t(x, y)$ defined by
$$\phi_t(x, y) = (xe^{2t}, ye^{-3t})$$
defines a C^∞ flow on $M = \mathbb{R}^2$. Determine the vector field that generates this flow (called the *infinitesimal generator of the flow*) and show that it is ϕ invariant.

4. Let $SO(3)$ be the set of 3×3 orthogonal matrices with determinant $+1$. The tangent space of $SO(3)$ at the identity is given by the set of skew-symmetric matrices of the form
$$\widehat{\omega} = (\omega)^\wedge = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$
(we'll show this later in the course).
   (a) Show that if $v \in \mathbb{R}^3$, $\widehat{\omega} v = \omega \times v$, where $\times$ is the cross product in $\mathbb{R}^3$.
   (b) Show that the tangent space $T_R SO(3)$ consists of matrices of the form $\widehat{\omega} R$ where $\widehat{\omega}$ is skew-symmetric.
   (c) Show that the flow of a vector field $g(R) = \widehat{\omega} R$ is given by $\phi_t(R) = \exp(\widehat{\omega} t) R$ where exp is the matrix exponential.

5. In this problem we will explore the properties of the *Lie bracket* between two vector fields, $[X, Y] = XY - YX$.
   (a) Let X, Y be two smooth vector fields on M. Show that $XY : C^\infty(M) \to C^\infty(M)$ is not a vector field but that $XY - YX$ is.
   (b) Let $X, Y \in \mathcal{X}(\mathcal{M})$ and $f, g \in C^\infty(M)$. Prove that
$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$