CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 131

R. Murray | **Homework Set #7** | Issued: 11 Nov 2020
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Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. Show that for a unity feedback system it suffices to check only two transfer functions to determine internal stability. [10]

2. Let
$$\widehat{P}(s) = \frac{1}{10s+1} \qquad \widehat{C}(s) = k \qquad \widehat{F}(s) = 1.$$
Find the least positive gain k such that the following are all true:
   (a) The feedback system is internally stable
   (b) $|e(\infty)| \leq 0.1$ when $r(t)$ is the unit step and $n = d = 0$.
   (c) $\|y\|_\infty \leq 0.1$ for all $d(t)$ such that $\|d\|_2 \leq 1$ when $r = n = 0$.

3. Consider a linear input/output system Σ with a minimal realization given by (A, B, C, D) and let the associated transfer function be $H(s) = C(sI - A)^{-1}B + D$. For simplicity, you may also assume that the system is SISO.
   (a) Show that if the linear system $\dot{x} = Ax$ is asymptotically stable then the induced input/output norm of the system Σ is bounded.
   (b) Show that if a linear input/output system Σ has bounded induced input/output norm, then the linear system $\dot{x} = Ax$ is asymptotically stable.
   (c) Show that if a linear system is input/output stable then $\|H\|_\infty$ is bounded.
   (d) Show via example that $\|H\|_\infty$ being bounded is not a sufficient condition for stability of the underlying system.

4. Consider the linear system (7.20). Let $u = -Kx$ be a state feedback control law obtained by solving the linear quadratic regulator problem. Prove the inequality
$$\big(I + L(-i\omega)\big)^T Q_u \big(I + L(i\omega)\big) \geq Q_u,$$
where
$$K = Q_u^{-1} B^T S, \qquad L(s) = K(sI - A)^{-1}B.$$
(Hint: Use the Riccati equation (7.33), add and subtract the terms sS, multiply with $B^T(sI + A)^{-T}$ from the left and $(sI - A)^{-1}B$ from the right.)

   For single-input single-output systems this result implies that the Nyquist plot of the loop transfer function has the property $|1 + L(i\omega)| \geq 1$, from which it follows that the phase margin for a linear quadratic regulator is always greater than $60°$.