CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 131

R. Murray — **Homework Set #3** — Issued: 14 Oct 2020

Fall 2020 — Due: 21 Oct 2020

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. [Sontag 3.1.2/3.1.3] Prove the following statements:

   (a) If $(x,\sigma) \rightsquigarrow (z,\tau)$ and $(z,\tau) \rightsquigarrow (y,\mu)$, then $(x,\sigma) \rightsquigarrow (y,\mu)$.

   (b) If $(x,\sigma) \rightsquigarrow (y,\mu)$ and if $\sigma < \tau < \mu$, then there exists a $z \in \mathcal{X}$ such that $(x,\sigma) \rightsquigarrow (z,\tau)$ and $(z,\tau) \rightsquigarrow (y,\mu)$.

   (c) If $x \underset{T}{\rightsquigarrow} y$ for some $T > 0$ and if $0 < t < T$, then there is some $z \in \mathcal{X}$ such that $x \underset{t}{\rightsquigarrow} z$ and $z \underset{T-t}{\rightsquigarrow} y$.

2. Consider the double integrator system $\ddot{y} = u$. Use the controllability Gramian to compute an input that steers the system for the origin to a state x_f in time T. What happens as $T \to 0$ and as $T \to \infty$?

3. [FBS 7.2] Extend the argument in Section 7.1 in *Feedback Systems* to show that if a system is reachable from an initial state of zero, it is reachable from a nonzero initial state.

4. [FBS 7.9] Consider the system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x,$$

   with the control law

$$u = -k_1 x_1 - k_2 x_2 + k_\text{f} r.$$

   Compute the rank of the reachability matrix for the system and show that eigenvalues of the system cannot be assigned to arbitrary values.

5. [Sontag 3.3.4] Assume that the pair (A,B) is not controllable with $\dim R(A,B) = \operatorname{rank} W_\text{c} = r < n$. From Lemma 3.3.3, there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that the matrices $\tilde{A} := T^{-1}AT$ and $\tilde{B} := T^{-1}B$ have the block structure

$$\tilde{A} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

   where $A_1 \in \mathbb{R}^{r\times r}$ and $B_1 \in \mathbb{R}^{r\times m}$. Prove that (A_1, B_1) is itself a controllable pair.