CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 131

R. Murray | **Homework Set #2** | Issued: 7 Oct 2020
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Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. [FBS 6.2] Show that a signal $u(t)$ can be decomposed in terms of the impulse function $\delta(t)$ as [10 pts]

$$u(t) = \int_0^t \delta(t-\tau)u(\tau)\,d\tau$$

and use this decomposition plus the principle of superposition to show that the response of a linear, time-invariant system to an input $u(t)$ (assuming a zero initial condition) can be written as a convolution equation

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau,$$

where $h(t)$ is the impulse response of the system. (Hint: Use the definition of the Riemann integral.)

2. [FBS 6.6] Consider a linear system with a Jordan form that is non-diagonal. [10 pts]

   (a) Prove Proposition 6.3 in *Feedback Systems* by showing that if the system contains a real eigenvalue $\lambda = 0$ with a nontrivial Jordan block, then there exists an initial condition with a solution that grows in time.

   (b) Extend this argument to the case of complex eigenvalues with $\operatorname{Re}\lambda = 0$ by using the block Jordan form

$$J_i = \begin{pmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{pmatrix}.$$

3. [Based on MIT 6-241j, 2011, Exercise 11.2] Consider the system [5 pts]

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with an input $u(t)$ that is piecewise linear:

$$u(t) = u[k](1 + a[k](t - kT))$$

   (a) Show that the sampled state $x[k] = x(kT)$ is governed by a sampled-data state-space model of the form:

$$x[k+1] = Fx[k] + Gu[k]$$

   for matrices F and G that do not depend on t and determine these matrices in terms of A, B, and $a[k]$. (Hint: The result will involve the matrix exponential, e^{At}.)

(b) For $T = 1$, how are the eigenvalues and eigenvectors of F related to those of A?

4. Consider a stable linear time-invariant system. Assume that the system is initially at rest and let the input be $u = \sin \omega t$, where ω is much larger than the magnitudes of the eigenvalues of the dynamics matrix. Show that the output is approximately given by [10 pts]

$$y(t) \approx |G(i\omega)| \sin\big(\omega t + \arg G(i\omega)\big) + \frac{1}{\omega} h(t),$$

where $G(s)$ is the frequency response of the system and $h(t)$ its impulse response.

5. Consider a linear system $\dot{x} = Ax$ with the matrix A given by [15 pts]

$$A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$.

(a) Find the stable, unstable, and center subspaces E^s, E^u, and E^c for $\lambda_1 > 0$ and $\lambda_2 < 0$.

(b) Qualitatively sketch the phase portrait of the system:

i. For $\lambda_1, \lambda_2 > 0$

ii. For $\lambda_1, \lambda_2 < 0$

iii. For $\lambda_1 > 0$ and $\lambda_2 < 0$

(c) Compute the matrix exponential, e^{At} for the system for all $\lambda_1, \lambda_2 \in \mathbb{R}$.

(d) From part (a), verify that $\mathbb{R}^2 = E^s \oplus E^u \oplus E^c$, where $\oplus$ represents the direct-sum of the vector spaces. Also verify that these subspaces are invariant under e^{At}.

(e) Give an example of a *non-hyperbolic* (Definition 2.2 FBS2s) linear system ($\dot{x} = Ax + Bu$, $y = Cx$). For all bounded inputs to your system, is the output bounded? Prove or give a counter example.