CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 131

R. Murray | **Homework Set #6** | Issued: 6 Nov 2019
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Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. Consider a closed loop system with process dynamics and a PI controller modeled by

$$\frac{dy}{dt} + ay = bu, \qquad u = k_\mathrm{p}(r - y) + k_\mathrm{i}\int_0^t \bigl(r(\tau) - y(\tau)\bigr)\,d\tau,$$

where r is the reference, u is the control variable, and y is the process output.

(a) Derive a differential equation relating the output y to the reference r by direct manipulation of the equations and compute the transfer function $H_{yr}(s)$. Make the derivations both by direct manipulation of the differential equations and by polynomial algebra.

(b) Draw a block diagram of the system and derive the transfer functions of the process $P(s)$ and the controller $C(s)$.

(c) Use block diagram algebra to compute the transfer function from reference r to output y of the closed loop system and verify that your answer matches your answer in part (a).

2. Show that the transfer function of a system depends only on the dynamics in the reachable and observable subspace of the Kalman decomposition. (Hint: Consider the representation given by equation (8.20).)

3. Consider the linear state space system

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx.$$

(a) Show that the transfer function is

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n},$$

where the coefficients for the numerator polynomial are linear combinations of the *Markov parameters* CA^iB, $i = 0, \ldots, n-1$:

$$b_1 = CB, \quad b_2 = CAB + a_1 CB, \quad \ldots, \quad b_n = CA^{n-1}B + a_1 CA^{n-2}B + \cdots + a_{n-1}CB$$

and $\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_n$ is the characteristic polynomial for A.

(b) Compute the transfer function for a linear system in reachable canonical form and show that it matches the transfer function given above.

4. Consider a closed loop system of the form of Figure 9.6, with $F = 1$ and P and C having a pole/zero cancellation. Show that if each system is written in state space form, the resulting closed loop system is not reachable and not observable.

5. [Dullerud and Paganini, 2.19] Consider a transfer function $G(s)$ and let (A, B, C, D) be a realization: $G(s) = C(sI - A)^{-1}B + D$.

   (a) Prove that if (A, B, C, D) is a minimal realization for $G(s)$ then every eigenvalue of A must be a pole of $G(s)$.

   (b) Prove that if (A, B, C, D) is a non-minimal realization, then every eigenvalue of A is either a pole of $G(s)$ or corresponds to an eigenvector that is in the unreachable or unobservable subspace.

6. Complete the proof of Theorem 1 in DFT by showing that if the polynomial $N_P N_C + M_P M_C$ has a zero s_0 in the right half plane, then it is not possible for s_0 to also a zero of all nine numerators in equation (3.2). Hence if $N_P N_C + M_P M_C$ has an unstable zero (pole of the system), at least one of the 4 transfer functions used to check internal stability for the pair (P, C) will be unstable.