CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 131

R. Murray | **Homework Set #5** | Issued: 30 Oct 2019
Fall 2019 | | Due: 6 Nov 2019

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. [FBS 8.1] Consider the system given by

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, and $y \in \mathbb{R}^q$. Show that the states can be determined from the input u and the output y and their derivatives if the observability matrix W_o given by equation (8.4) has n independent rows.

2. Consider a n-dimensional linear discrete time system with p inputs and q outputs

$$x[k+1] = Ax[k] + Bu[k], \qquad y[k] = Cx[k],$$

where $x[0] = x_0$ is the unknown initial condition, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{q\times n}$. Prove that the system is observable if and only if the $n \times n$ observability Gramian for the discrete-time system has full rank:

$$W_\text{o}[n] = \sum_{i=0}^{n-1} (A^T)^i C^T C A^i.$$

Describe how the observability condition is different for a discrete-time system from the continuous-time dynamics studied in class (and the notes).

3. Show that the set of unobservable states for a linear system with dynamics matrix A and output matrix C is an A-invariant subspace and that it is equal to the largest A-invariant subspace annihilated by C.

4. [FBS 8.4] Show that if a system is observable, then there exists a change of coordinates $z = Tx$ that puts the transformed system into observable canonical form.

5. [FBS 8.15] Consider a linear system characterized by the matrices

$$A = \begin{pmatrix} -2 & 1 & -1 & 2 \\ 1 & -3 & 0 & 2 \\ 1 & 1 & -4 & 2 \\ 0 & 1 & -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}, \quad D = 0.$$

Construct a Kalman decomposition for the system. (Hint: Try to diagonalize.)

6. Consider a control system having state space dynamics

$$\frac{dx}{dt} = \begin{bmatrix} -\alpha-\beta & 1 \\ -\alpha\beta & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ k \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

(a) Construct an observer for the system and find expressions for the observer gain $L = \begin{pmatrix} l_1 & l_2 \end{pmatrix}^T$ such that the observer has natural frequency ω_0 and damping ratio ζ.

(b) Suppose that we choose a different output

$$\tilde{y} = \begin{bmatrix} 1 & \gamma \end{bmatrix} x.$$

Are there any values of γ for which the system is *not* observable? If so, provide an example of an initial condition and output where it is not possible to uniquely determine the state of the system by observing its inputs and outputs.