CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 131

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Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. [OBC 2.1]

   (a) Let $G_1, G_2, \ldots, G_k$ be a set of row vectors on a $\mathbb{R}^n$. Let F be another row vector on $\mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$ satisfying $G_i x = 0$, $i = 1, \ldots, k$, we have $Fx = 0$. Show that there are constants $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that

   $$F = \sum_{i=1}^{k} \lambda_i G_i.$$

   (b) Let $x^* \in \mathbb{R}^n$ be an the extremal point (maximum or minimum) of a function f subject to the constraints $g_i(x) = 0$, $i = 1, \ldots, k$. Assuming that the gradients $\partial g_i(x^*)/\partial x$ are linearly independent, show that there are k scalers λ_i, $i = 1, \ldots, k$ such that the function

   $$\tilde{f}(x) = f(x) + \sum_{i=1}^{k} \lambda_i g_i(x)$$

   has an extremal point at x^*.

2. [FBS 7.19] Use the Riccati equation (7.31) and the relation

   $$x^T(t_\text{f}) Q_\text{f} x(t_\text{f}) - x^T(0) S(0) x(0) = \int_0^{t_\text{f}} \Big(\dot{x}^T(t) S(t) x(t) + x^T \dot{S}(t) x(t) + x^T(t) S(t) \dot{x}(t) \Big) dt.$$

   to show that the cost function for the linear quadratic regulator problem can be written as

   $$\int_0^{t_\text{f}} \Big(x^T(t) Q_x x(t) + u^T(t) Q_u u(t) \Big) dt + x^T(t_\text{f}) Q_\text{f} x(t_\text{f}) = x^T(0) S(0) x(0) + \int_0^{t_\text{f}} \Big(u(t) + Q_u^{-1} B^T S(t) x(t) \Big)^T Q_u \Big(u(t) + Q_u^{-1} B^T S(t) x(t) \Big) dt,$$

   from which it follows that the control law $u(t) = -Kx(t) = -Q_u^{-1} B^T S(t) x(t)$ is optimal. Does the proof hold when all matrices depend on time?

3. [FBS 7.21] Consider the Riccati equation

   $$-\frac{dS}{dt} = A^T S + SA - SBQ_u^{-1} B^T S + Q_x, \qquad S(t_f) = Q_\text{f},$$

which is quadratic in S. Show that the solution is

$$S(t) = [\Psi_{21}(t) + \Psi_{22}(t)Q_{\mathrm{f}}][\Psi_{11}(t) + \Psi_{12}(t)Q_{\mathrm{f}}]^{-1},$$

where the matrix Ψ satisfies the (linear) differential equation

$$\frac{d\Psi}{dt} = \frac{d}{dt}\begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} = \begin{pmatrix} A & -BQ_u^{-1}B^T \\ -Q_x & -A^T \end{pmatrix} \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix},$$

with initial conditions

$$\Psi(t_f) = \begin{pmatrix} \Psi_{11}(t_f) & \Psi_{12}(t_f) \\ \Psi_{21}(t_f) & \Psi_{22}(t_f) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

4. Consider the double integrator

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Ax + Bu.$$

Find a state feedback that minimizes the quadratic cost function

$$J = \int_0^\infty \left(q_1 x_1^2 + q_2 x_2^2 + q_u u^2\right) dt$$

where $q_1 \geq 0$ is the penalty on position, $q_2 \geq 0$ is the penalty on velocity, and $q_u > 0$ is the penalty on control actions. Analyze the coefficients of the closed loop characteristic polynomial and explore how they depend on the penalties.