CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 131

Homework Set #2

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Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. [FBS 6.1] Show that if $y(t)$ is the output of a linear time-invariant system corresponding to input $u(t)$, then the output corresponding to an input $\dot{u}(t)$ is given by $\dot{y}(t)$. (Hint: Use the definition of the derivative: $\dot{z}(t) = \lim_{\epsilon\to 0}\bigl(z(t+\epsilon) - z(t)\bigr)/\epsilon$.)

2. [FBS 6.2] Show that a signal $u(t)$ can be decomposed in terms of the impulse function $\delta(t)$ as

$$u(t) = \int_0^t \delta(t-\tau)u(\tau)\,d\tau$$

and use this decomposition plus the principle of superposition to show that the response of a linear, time-invariant system to an input $u(t)$ (assuming a zero initial condition) can be written as a convolution equation

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau,$$

where $h(t)$ is the impulse response of the system. (Hint: Use the definition of the Riemann integral.)

3. [FBS 6.4] Assume that $\zeta < 1$ and let $\omega_\text{d} = \omega_0\sqrt{1-\zeta^2}$. Show that

$$\exp\begin{pmatrix} -\zeta\omega_0 & \omega_\text{d} \\ -\omega_\text{d} & -\zeta\omega_0 \end{pmatrix} t = e^{-\zeta\omega_0 t}\begin{pmatrix} \cos\omega_\text{d} t & \sin\omega_\text{d} t \\ -\sin\omega_\text{d} t & \cos\omega_\text{d} t \end{pmatrix}.$$

Also show that

$$\exp\left(\begin{pmatrix} -\omega_0 & \omega_0 \\ 0 & -\omega_0 \end{pmatrix} t\right) = e^{-\omega_0 t}\begin{pmatrix} 1 & \omega_0 t \\ 0 & 1 \end{pmatrix}.$$

4. [FBS 6.6] Consider a linear system with a Jordan form that is non-diagonal.

   (a) Prove Proposition 6.3 in *Feedback Systems* by showing that if the system contains a real eigenvalue $\lambda = 0$ with a nontrivial Jordan block, then there exists an initial condition with a solution that grows in time.

   (b) Extend this argument to the case of complex eigenvalues with $\operatorname{Re}\lambda = 0$ by using the block Jordan form

$$J_i = \begin{pmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{pmatrix}.$$

5. [FBS 6.8] Consider a linear discrete-time system of the form

$$x[k+1] = Ax[k] + Bu[k], \qquad y[k] \ = Cx[k] + Du[k].$$

(a) Show that the general form of the output of a discrete-time linear system is given by the discrete-time convolution equation:

$$y[k] = CA^k x[0] + \sum_{j=0}^{k-1} CA^{k-j-1} Bu[j] + Du[k].$$

(b) Show that a discrete-time linear system is asymptotically stable if and only if all the eigenvalues of A have a magnitude strictly less than 1.

6. Consider a linear system $\dot{x} = Ax$ with the matrix A given by

$$A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$.

(a) Find the stable, unstable, and center subspaces E^s, E^u, and E^c for $\lambda_1 > 0$ and $\lambda_2 < 0$.

(b) Qualitatively sketch the phase portrait of the system:

i. For $\lambda_1, \lambda_2 > 0$

ii. For $\lambda_1, \lambda_2 < 0$

iii. For $\lambda_1 > 0$ and $\lambda_2 < 0$

(c) Compute the matrix exponential, e^{At} for the system for all $\lambda_1, \lambda_2 \in \mathbb{R}$.

(d) From part (a), verify that $\mathbb{R}^2 = E^s \oplus E^u \oplus E^c$, where $\oplus$ represents the direct-sum of the vector spaces. Also verify that these subspaces are invariant under e^{At}.

(e) Give an example of a *non-hyperbolic* (Definition 2.2 FBS2s) linear system ($\dot{x} = Ax$, $y = Cx$). For all bounded inputs to your system, is the output bounded? Prove or give a counter example.