CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 131

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Homework Set #1

Issued: 2 Oct 2019
Due: 9 Oct 2019

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. Suppose that $u(t)$ is a continuous signal whose derivative $\dot{u}(t)$ is also continuous. Which of the following quantities qualifies as a norm for u:

   (a) $\sup_t |\dot{u}(t)|$

   (b) $|u(0)| + \sup_t |\dot{u}(t)|$

   (c) $\max\{\sup_t |u(t)|, \sup_t |\dot{u}(t)|\}$

   (d) $\sup_t |u(t)| + \sup_t |\dot{u}(t)|$

   Make sure to give a thorough answer (not just yes or no).

2. Consider a discrete time system having dynamics

$$x[k+1] = Ax[k] + Bu[k], \qquad y[k] = Cx[k],$$

   where $x[k] \in \mathbb{R}^n$ is the state of the system at time $k \in \mathbb{Z}$, $u[k] \in \mathbb{R}$ is the (scalar) input for the system, $y[k] \in \mathbb{R}$ is the (scalar) output for the system and A, B, and C are constant matrices of the appropriate size. We use the notation $x[k] = x(kh)$ to represent the state of the system at discrete time k where $h \in \mathbb{R}$ is the sampling time (and similarly for $u[k]$ and $y[k]$).

   Let $\mathcal{T} = [0, h, \ldots, Nh]$ represent a discrete time range, with $N \in \mathbb{Z}$.

   (a) Considered as a dynamical system over $\mathcal{T}$, what is the input space $\mathcal{U}$, output space $\mathcal{Y}$, and state space Σ corresponding to the dynamics above? Show that each of these spaces is a linear space by verifying the required properties (you may assume that $\mathbb{R}^p$ is a linear space for appropriate p).

   (b) What is the state transition function $s(t_1, t_0, x_0, u(\,\cdot\,))$? Show that this function satisfies the state transition axiom and the semi-group axiom.

   (c) What is the readout function $r(t, x, u)$? Show that the input/output system is a linear input/output dynamical system over $\mathcal{T}$.

   (d) What is the zero-input response for the system? What is the zero-state response for the system?

3. Let D be a pure time delay of τ seconds with transfer function

$$\widehat{D}(s) = e^{-s\tau}.$$

A norm $\|\cdot\|$ on transfer functions is *time-delay invariant* if for every bounded transfer function $\widehat{G}$ and every $\tau > 0$ we have

$$\|\widehat{D}\widehat{G}\| = \|\widehat{G}\|$$

Determine if the 2-norm and ∞-norm are time-delay invariant.

4. Let $\widehat{G}$ be the transfer function for a stable, proper plant (but not necessarily strictly proper).

   (a) Show that the ∞-norm of the output y given an input $u(t) = \sin(\omega t)$ is $|\widehat{G}(jw)|$.

   (b) Show that the 2-norm to 2-norm system gain for $\widehat{G}$ is $\|\widehat{G}\|_\infty$ (just as in the strictly proper case).

5. For a linear system with input u and output y, prove that

$$\sup_{\|u\| \leq 1} \|y\| = \sup_{\|u\|=1} \|y\|$$

where $\|\cdot\|$ is any norm on signals.