Feedback Systems:
Notes on Linear Systems Theory

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These notes are a supplement for the second edition of *Feedback Systems* by Åström and Murray (referred to as FBS2e), focused on providing some additional mathematical background and theory for the study of linear systems.

Contents

Chapter 1

Signals and Systems

The study of linear systems builds on the concept of linear maps over vector spaces, with inputs and outputs represented as function of time and linear systems represented as a linear map over functions. In this chapter we review the basic concepts of linear operators over (infinite-dimensional) vector spaces, define the notation of a linear system, and define metrics on signal spaces that can be used to determine norms for a linear system. We assume a basic background in linear algebra.

1.1 Linear Spaces and Mappings

We briefly review here the basic definitions for linear spaces, being careful to take a general view that will allow the underlying space to be a signal space (as opposed to a finite dimensional linear space).

Definition 1.1. A set V is a *linear space over* $\mathbb{R}$ if the following axioms hold:

1. Addition: For every $x, y \in V$ there is a unique element $x + y \in V$ where the addition operator $+$ satisfies:
   (a) Commutativity: $x + y = y + x$.
   (b) Associativity: $(x + y) + z = x + (y + z)$.
   (c) Additive identity element: there exists an element $0 \in V$ such that $x + 0 = x$ for all $x \in V$.
   (d) Additive inverse: For every $x \in V$ there exists a unique element $-x \in V$ such that $x + (-x) = 0$.
2. Scalar multiplication: For every $\alpha \in \mathbb{R}$ and $x \in V$ there exists a unique vector $\alpha x \in V$ and the scaling operator satisfies:
   (a) Associativity: $(\alpha\beta) = \alpha(\beta x)$.
   (b) Distributivity over addition in V: $\alpha(x + y) = \alpha x + \alpha y$.
   (c) Distributivity over addition in $\mathbb{R}$: $(\alpha + \beta)x = \alpha x + \beta x$.
   (d) Multiplicative identity: $1 \cdot x = x$ for all $x \in V$.

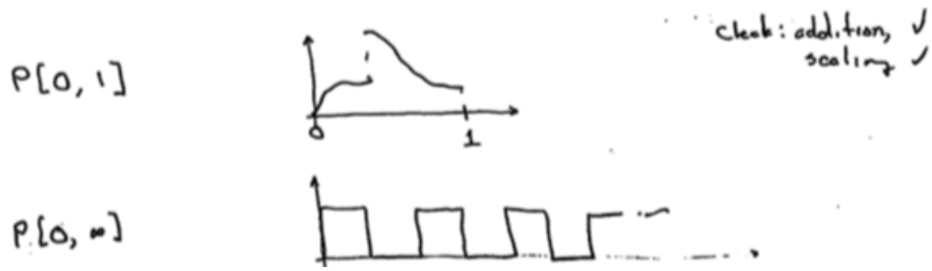
More generally, we can replace $\mathbb{R}$ with any *field* (such as complex number $\mathbb{C}$). The terms "vector space", "linear space", and "linear vector space" will be used interchangeably throughout the text.

A vector space V is said to have a basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ is any element $v \in V$ can be written as a linear combination of the basis vectors v_i and the elements of $\mathcal{B}$ are linearly independent. If such a basis exists for a finite n, then V is said to be finite-dimensional of dimension n. If no such basis exists for any finite n then the vector space is said to be infinite-dimensional.

Example 1.1 ($\mathbb{R}^n$)**.** The finite-dimensional vector space $V = \mathbb{R}^n$ consisting of elements $x = (x_1, \ldots, x_n)$ is a vector space over the reals, with the addition and scaling operations defined as

$$x + y = (x_1 + y_1, \ldots, x_n + y_n)$$
$$\alpha x = (\alpha x_1, \ldots, \alpha_n)$$

Example 1.2 ($\mathcal{P}[t_0, t_1]$)**.** The space of piecewise continuous mappings from a time interval $[t_0, t_1] \subset \mathbb{R}$ to $\mathbb{R}$ is defined as the set of functions $F : [t_0, t_1] \to \mathbb{R}$ that have a finite set of discontinuities on every bounded subinterval.



As an exercise, the reader should verify that the axioms of a linear space are satisfied.

Extensions and special cases include:

1. $\mathcal{P}^n[t_0, t_1]$: the space of piecewise continuous functions taking values in $\mathbb{R}^n$.

2. $\mathcal{C}^n[t_0, t_1]$: the space of continuous functions $F : [t_0, t_1] \to \mathbb{R}^n$.

All of these vector spaces are infinite dimensional.

Example 1.3 ($V_1 \times V_2$)**.** Given two linear spaces V_1 and V_2 of the same type, the Cartesian product $V_1 \times V_2$ is a linear space with addition and scaling defined component-wise. For example, $\mathbb{R}^n \times \mathbb{R}^m$ is the linear space $\mathbb{R}^{m+n}$ and the linear space $\mathcal{C}[t_0, t_1] \times \mathcal{C}[t_0, t_1]$ is a linear space $\mathcal{C}^2[t_0, t_1]$ with the operations

$$(f, g)(t) = (f(t), g(t)), \tag{S1.1}$$
$$(f_1, g_1) + (f_2, g_2) = (f_1 + g_1, f_2 + g_2), \tag{S1.2}$$
$$\alpha(f, g) = (\alpha f, \alpha g). \tag{S1.3}$$

Given a vector space V over the reals, we can define a *norm* on the vector space that associates with each element $x \in V$ a real number $\|x\| \in \mathbb{R}$.

Definition 1.2. A mapping $\|\cdot\| : V \to \mathbb{R}$ is a *norm* on V if it satisfies the following axioms:

1. $\|x\| \geq 0$ for all $x \in V$.

2. $\|x\| = 0$ if and only if $x = 0$.
3. $\|\alpha x\| = |\alpha|\,\|x\|$ for all $x \in V$ and $\alpha \in \mathbb{R}$.
4. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$ (called the *triangle inequality*).

These definitions are easy to verify for finite-dimensional vector spaces, but they hold even if a vector space is infinite-dimensional.

The following table describes some standard norms for finite-dimensional and infinite dimensional linear spaces.

Name	$V = \mathbb{R}^n$	$V = \{\mathbb{Z}_+ \to \mathbb{R}^n\}$	$V = \{(-\infty, \infty) \to \mathbb{R}\}$
1-norm, $\|\cdot\|_1$	$\sum_i \lvert x_i \rvert$	$\sum_k \|x[k]\|$	$\int_{-\infty}^{\infty} \lvert u(\tau) \rvert, d\tau$
2-norm, $\|\cdot\|_2$	$\sqrt{\sum_i \lvert x_i \rvert^2}$	$\left(\sum_k \|x[k]\|^2\right)^{1/2}$	$\left(\int_{-\infty}^{\infty} \lvert u(\tau) \rvert^2, d\tau\right)^{1/2}$
p-norm, $\|\cdot\|_p$	$\sqrt[p]{\sum_i \lvert x_i \rvert^p}$	$\left(\sum_k \|x[k]\|^2\right)^{1/p}$	$\left(\int_{-\infty}^{\infty} \lvert u(\tau) \rvert^p, d\tau\right)^{1/p}$
∞-norm, $\|\cdot\|_\infty$	$\max_i \lvert x_i \rvert$	$\max_k \|x[k]\|$	$\sup_t \lvert u(t) \rvert$

(The function sup is the supremum, where $\sup_t u(t)$ is the smallest number $\bar{u}$ such that $u(t) \leq \bar{u}$ for all t.)

A linear space equipped with a norm is called a *normed linear space*. A normed linear space is said to be *complete* if every Cauchy sequence in V converges to a point in V. (A sequence $\{x_i\}$ is a Cauchy sequence if for every $\epsilon > 0$ there exists an integer N such that $\|x_p - x_q\| < \epsilon$ for all $p, q > N$.) Not every normed linear space is complete. For example, the normed linear space $\mathcal{C}[0, \infty)$, consisting of continuous, real-valued functions is not complete since it is possible to construct a sequence of continuous functions that converge to a discontinuous function (for example a step function). The space $\mathcal{P}[0, \infty)$ consisting of piecewise continuous functions is complete. A complete normed linear space is called a *Banach space*.

Let V and W be linear spaces over $\mathbb{R}$ (or any common field). A mapping $A : V \to W$ is a linear map if

$$A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha A v_1 + \alpha_2 V_2$$

for all $\alpha_1, \alpha_2 \in \mathbb{R}$ and $v_1, v_2 \in V$. Examples include:

1. Matrix multiplication on $\mathbb{R}^n$.
2. Integration operators on $\mathcal{P}[0, 1]$: $Av = \int_0^1 v(t)\, dt$.
3. Convolution operators: let $h \in \mathcal{P}[0, \infty)$ and define the linear operator C_h as

$$(C_h v)(t) = \int_0^t h(t - \tau) v(\tau)\, d\tau$$

This last item provides a hint of how we will define a linear system.

Definition 1.3. An *inner product* on a linear space V is a mapping $\langle \cdot\,, \cdot \rangle : V \times V \to \mathbb{R}$ with the following properties:

1. Bilinear: $\langle \alpha_1 v_1 + \alpha_2 v_2, w \rangle = \alpha_1 \langle v_1, w \rangle + \alpha_2 \langle v_2, w \rangle$ and the same for the second argument.

2. Symmetric: $\langle v, w\rangle = \langle w, v\rangle$
3. Positive definite: $\langle v, v\rangle > 0$ if $v \neq 0$.

A (complete) linear space with an inner product is called a *Hilbert* space. The inner produce also defines a norm given by $\|v\| = \langle v, v\rangle$. A property of the inner product is that $|\langle u, v\rangle| \leq \|u\|_2 \cdot \|v\|_2$ (the Cauchy-Schwartz inequality), which we leave as an exercise (hint: rewrite u as $u = z + (\langle u, v\rangle/\|v\|)v$ where z can be shown to be orthogonal to u).

Example 1.4 (2-norm)**.** Let $V = \mathcal{C}(-\infty, \infty)$. Then $\|\cdot\|_2$ can be verified to be a norm by checking each of the axioms:

1. $\|u\|_2 = \left(\int_{-\infty}^{\infty} |u(t)|^2\, dt\right)^{1/2} > 0$.
2. If $u(t) = 0$ for all t then $\|u\|_2 = 0$ by definition. To see the converse, assume that $\|u\|_2 = 0$. Then by definition we must have
$$\int_{-\infty}^{\infty} |u(t)|^2\, dt = 0$$
and therefore $\|u\|_2 = 0$ on any subset of $(-\infty, \infty)$. Since $\mathcal{C}(-\infty, \infty)$ consists of continuous functions, it follows that $u(t) = 0$ at all points t (if not, then there would be a subset of $(-\infty, \infty)$ on which $|u(t)| > 0$ and the integral would not be zero.
3. $\|\alpha u\|_2 = \left(\int_{-\infty}^{\infty} |\alpha u(t)|^2\, dt\right)^{1/2} = \alpha\|u\|_2$.
4. To show the triangle inequality for the 2-norm, we make use of the Cauchy-Schwartz inequality by defining the inner product between two elements of V as
$$\langle u, v\rangle = \int_{-\infty}^{\infty} u(t)v(t)\, dt.$$
It can be shown that this satisfies the properties of an inner product. Using the fact that $\|u\|_2 = \langle u, u\rangle$ we can show that
$$\begin{aligned}\|u+v\|_2^2 &= \int_{-\infty}^{\infty} |u(t)|^2 + 2u(t)v(t) + |v(t)|^2\, dt \\ &= \|u\|_2^2 + 2\langle u(t), v(t)\rangle\, dt + \|v\|_2^2 \\ &\leq \|u\|_2^2 + 2|\langle u(t), v(t)\rangle|\, dt + \|v\|_2^2 \\ &\leq \|u\|_2^2 + 2\|u\|_2 \cdot \|v\|_2 + \|v\|_2^2 = (\|u\|_2 + \|v\|_2)^2\end{aligned}$$

1.2 Input/Output Dynamical Systems

We now proceed to define an input/output dynamical system, with an eventual focus on linear input/output dynamical systems. It is useful to distinguish between three different conceptual aspects of a "dynamical system:

- A *physical system* represents a physical (or biological or chemical) system that we are trying to analyze or design. An example of a physical system would be a vectored thrust aircraft or perhaps a laboratory experiment intended to test different control algorithms.

- A *system model* is an idealized version of the physical system. There may be many different system models for a given physical system, depending on what questions we are trying to answer. A model for a vectored thrust aircraft might be a simplified, planar version of the system (relevant for understanding basic tradeoffs), a nonlinear model that takes into account actuation and sensing characteristics (relevant for designing controllers that would be implemented on the physical system), or a complex model including bending modes, thermal properties and other details (relevant for doing model-based assessment of complex specifications involving those attributes).

- A *system representation* is a mathematical description of the system using one or more mathematical frameworks (e.g., ODEs, PDEs, automata, etc).

In the material that follows, we will use the word "system" to refer to the system representation, but keeping in mind that this is just a mathematical abstraction of a system model that is itself an approximation of the actual physical system.

Definition 1.4. Let $\mathcal{T}$ be a subset of $\mathbb{R}$ (usually $\mathcal{T} = [0, \infty)$ or $\mathcal{T} = \mathbb{Z}_+$). A *dynamical system* on $\mathcal{T}$ is a representation consisting of a tuple $\mathcal{D} = (\mathcal{U}, \Sigma, \mathcal{Y}, s, r)$ where

- the *input space* $\mathcal{U}$ is a set of functions mapping $\mathcal{T}$ to a set U representing the set of possible inputs to the system (typically $\mathcal{U} = \mathcal{P}^m[0, \infty)$);

- the *state space* Σ is a set representing the state of the system (usually $\mathbb{R}^n$, but can also be infinite dimensional, for example when time delays or partial differential equations are used);

- the *output space* $\mathcal{Y}$ is set of functions mapping $\mathcal{T}$ to a set Y representing the set of measured outputs of the system (typically $\mathcal{Y} = \mathcal{P}^p[0, \infty)$);

- the *state transition function* $s : \mathcal{T} \times \mathcal{T} \times \Sigma \times \mathcal{U} \to \Sigma$ is a function of the form $s(t_1, t_0, x_0, u(\,\cdot\,))$ that returns the state $x(t_1)$ of the system at time t_1 reached from state x_0 at time t_0 as a result of applying an input $u \in \mathcal{U}$;

- the *readout function* $r : \mathcal{T} \times \Sigma \times U \to Y$ is a function of the form $r(t, x, u)$ that returns the output $y(t) \in Y$ representing the value of the measured outputs of the system at time $t \in \mathcal{T}$ given that we are at state $x \in \Sigma$ and applying input $u \in U$.

Furthermore, the following axioms must be satisfied:

(A1) State transition axiom: for any $t_0, t_1 \in \mathcal{T}$ and $x_0 \in \Sigma$ with $t_1 \geq t_0$, if $u(\,\cdot\,), \tilde{u}(\,\cdot\,) \in \mathcal{U}$ and

$$u(t) = \tilde{u}(t) \quad \text{for all} \quad t \in [t_0, t_1] \cap \mathcal{T}$$

then

$$s(t_1, t_0, x_0, u(\,\cdot\,)) = s(t_1, t_0, x_0, \tilde{u}(\,\cdot\,)).$$

(A2) Semi-group axiom: For all $t_0 \leq t_1 \leq t_2 \in \mathcal{T}$, all $x_0 \in \Sigma$, and all $u(\,\cdot\,) \in \mathcal{U}$

$$s(t_2, t_1, s(t_1, t_0, x_0, u(\,\cdot\,)), u(\,\cdot\,) = s(t_2, t_0, x_0, u(\,\cdot\,)).$$

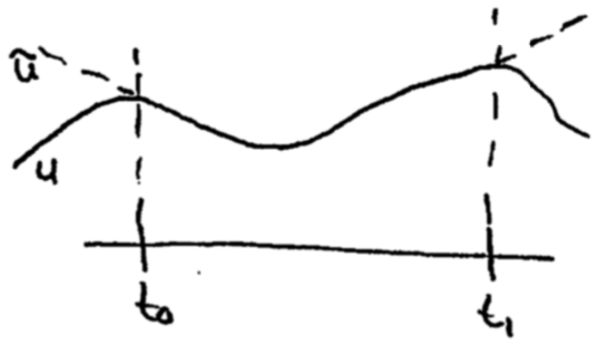


Figure S1.1: Illustration of the state transition axiom.

The definition of a dynamical system captures precisely the notion of a system that has an internal "state" $x \in \Sigma$ and that this state summarizes all information about the system at a given time. Axiom A1 states that inputs differ before reaching a state x_0 and after reaching a state x_1 but are otherwise the same will generate the same trajectory in state space, as illustrated in Figure S1.1. Axiom A2 has the interpretation that we can compute the state at time t_2 by first calculating the state at some intermediate time t_1. In both cases, these are formal statements that the state $x(t)$ summarizes all effects due to the input prior to time t.

Example 1.5 (Input/output differential equation representation)**.** A nonlinear input/output system can be represented as the differential equation

$$\frac{dx}{dt} = f(x, u), \qquad y = h(x, u), \tag{S1.4}$$

where x is a vector of state variables, u is a vector of control signals, and y is a vector of measurements. The term dx/dt represents the derivative of the vector x with respect to time, and f and h are (possibly nonlinear) mappings of their arguments to vectors of the appropriate dimension.

For mechanical systems, the state consists of the configuration variables $q \in \mathbb{R}^n$ and time derivatives of the configuration variables $\dot{q} \in \mathbb{R}^n$ (representing the generalized velocity of the system), so that $x = (q, \dot{q}) \in \mathbb{R}^{2n}$. Note that in the dynamical system formulation of mechanical systems we model the dynamics as first-order differential equations, rather than the more traditional second-order form (e.g., Lagrange's equations), but it can be shown that first order differential equations can capture the dynamics of higher-order differential equations by appropriate definition of the state and the maps f and h.

A model is called a *linear* state space model if the functions f and h are linear in x and u. A linear state space model can thus be represented by

$$\frac{dx}{dt} = A(t)x + B(t)u, \qquad y = C(t)x + D(t)u, \tag{S1.5}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are constant matrices of appropriate dimension. The matrix A is called the *dynamics matrix*, the matrix B is called the *control matrix*, the matrix C is called the *sensor matrix*, and the matrix D is called the *direct term*. Frequently models will not have a direct term, indicating that the input signal u does not influence the output directly.

This definition of a dynamical system is not the most general one possible. In particular, we note that our definition is restricted to model systems that are *causal*: the current state depends only on the past inputs. Furthermore, we have ignored the important class of stochastic dynamical systems, in which the inputs, outputs, and states are described by probability distributions rather than deterministic values. Similarly, this class of systems does not capture other types of non-deterministic systems where a single state may lead to more than one possible output, a situation that is not uncommon in automata theory.

In addition to restricting ourselves to deterministic, causal dynamical systems, we will also often be interested in the case where the system is time-invariant as well. To define time invariance we define the *shift operator* $T_\tau : \mathcal{U} \to \mathcal{U}$ as $(T_\tau u)(t) = u(t+\tau)$. We further define the *input/output map* $\rho : \mathcal{T} \times \mathcal{T} \times \Sigma \times \mathcal{U} \to Y$ as

$$\rho(t, t_0, x_0, u(\,\cdot\,)) = r(t, s(t, t_0, x_0, u(\,\cdot\,)), u(t)),$$

which allows us to evaluate the output of the system at time t given the initial state $x(t_0) = x_0$ and the input applied to the system.

Definition 1.5. A dynamical systems is *time invariant* if

1. $\mathcal{U}$ is closed under translation:

$$u(\,\cdot\,) \in \mathcal{U} \quad \Longrightarrow \quad T_\tau u(\,\cdot\,) \in \mathcal{U}.$$

2. The input/output map is *shift invariant*:

$$\rho(t_1, t_0, x_0, u(\,\cdot\,)) = \rho(t_1 + \tau, t_0 + \tau, x_0, T_\tau u(\,\cdot\,)).$$

It is straightforward to show that a linear state space model is time invariant if the matrices $A(t)$, $B(t)$, $C(t)$, and $D(t)$ do not depend on time, leading to the representation

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du. \tag{S1.6}$$

For our purposes, we will use a slightly more general description of a linear dynamical system, focusing on input/output properties.

Definition 1.6. An input/output dynamical system is a *linear input/output dynamical system* if

1. $\mathcal{U}$, Σ, and $\mathcal{Y}$ are linear spaces over $\mathbb{R}$ (or some other common field, such as $\mathbb{C}$);
2. for fixed $t, t_0 \in \mathcal{T}$ with $t \geq t_0$, $\rho : \mathcal{T} \times \mathcal{T} \times \Sigma \times \mathcal{U} \to Y$ is linear in $\Sigma \times \mathcal{U}$ onto Y:

$$\begin{aligned} \rho(t, t_0, x_0, u(\,\cdot\,)) &= \rho(t, t_0, x_0, 0) + \rho(t, t_0, 0, u(\,\cdot\,)) \\ \rho(t, t_0, \alpha x + \beta x', 0) &= \alpha \rho(t, t_0, x, 0) + \beta \rho(t, t_0, x', 0) \\ \rho(t, t_0, 0, \alpha u(\,\cdot\,) + \beta u'(\,\cdot\,)) &= \alpha \rho(t, t_0, 0, u(\,\cdot\,)) + \beta \rho(t, t_0, 0, u'(\,\cdot\,)). \end{aligned}$$

It follows from this definition that if $\mathcal{D}$ is a linear dynamical system representation then the output response can be divided into an initial condition (zero-input) response and a force (zero-initial state) response:

$$\rho(t, t_0, x_0, u(\,\cdot\,)) = \underbrace{\rho(t, t_0, x_0, 0)}_{\text{zero-input response}} + \underbrace{\rho(t, t_0, 0, u(\,\cdot\,))}_{\text{zero-state response}}.$$

Furthermore, the principle of superposition holds for the zero-state response:

$$\rho(t, t_0, x_0, \alpha u(\,\cdot\,) + \beta u'(\,\cdot\,)) = \alpha\rho(t, t_0, x_0, u(\,\cdot\,)) + \beta\rho(t, t_0, x_0, u(\,\cdot\,)).$$

These properties will be familiar to readers who have already encountered linear input/output systems in signal processing or control theory, though we do note here the subtlety that these definitions and properties hold in the time-varying case as well as for time-invariant systems.

For the remainder of the notes we will restrict ourselves to linear, time-invariant (LTI) representations. We will also generally concentrate on the zero-state response, corresponding to the (pure) input/output response.

1.3 Linear Systems and Transfer Functions

Let G be a linear, time-invariant, causal, finite-dimensional system. A different way of defining G is to define the zero-state response as a *convolution equation*:

$$y = G * u, \qquad y(t) = \int_{-\infty}^{\infty} G(t-\tau)u(\tau)\,d\tau.$$

In this formulation, the function $G : (-\infty, \infty) \to \mathbb{R}^m$ is called the *impulse* response of the system and can be regarding as the response of the system to a unit impulse $\delta(t)$ (see FBS2e for the definition of the impulse function). The term $G(t-\tau)$ then represents the response of the system at time t to an input and time τ and the convolution equation is constructed by considering the input to be the convolution of the impulse function $\delta(\,\cdot\,)$ with the input $u(\,\cdot\,)$ and applying the principle of superposition. We also note that if the system is causal then $G(t) = 0$ for all $t < 0$ (if this is not the case, then $y(t)$ and depending on $u(\tau)$ for $\tau < t$).

An alternative to representation of the input/output response as a convolution integral is to make use of the (one-sided) Laplace transform of the inputs, outputs, and impulse response. Letting $\hat{Y}(s)$ represent the Laplace transform of the signal $y(t)$w where $s \in \mathbb{C}$ is the Laplace variable, we have

$$\begin{aligned}
\hat{Y}(s) &= \int_0^{\infty} y(t)e^{-st}\,dt \\
&= \int_0^{\infty} \left(\int_0^{\infty} G(t-\tau)u(\tau)\,d\tau\right) e^{-st}\,dt \\
&= \int_0^{\infty}\int_0^{\infty} \left(G(t-\tau)u(\tau)\,e^{-s(t-\tau)}\,dt\right)\,d\tau \\
&= \underbrace{\left(\int_0^{\infty} G(t)e^{-st}\,dt\right)}_{\hat{G}(s)} \underbrace{\left(\int_0^{\infty} u(\tau)e^{-s\tau}\,d\tau\right)}_{\hat{U}(s)}.
\end{aligned}$$

The Laplace transform of $y(t)$ is thus given by the product of the Laplace transform of the impulse response $G(t)$ and the Laplace transform of the input $u(t)$. The function $\hat{G}(s)$ is called the *transfer function* between input u and output y and represents the zero-state, input/output response of the system. Notationally, we will often write $\hat{G}_{yu}$ to represent the transfer function from u to y so that we have

$$\hat{Y}(s) = \hat{G}_{yu}(s)\hat{U}(s).$$

For a system with m inputs and p outputs, a transfer function $\hat{G}(s)$ represents a mapping from $\mathbb{C}$ to $\mathbb{R}^{p\times m}$. Similar to our definition of norms for signal spaces, we can define norms for Laplace transforms. For the single-input, single-output (SISO) case we define

$$\|\hat{G}\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2\, d\omega\right)^{1/2}, \qquad \|\hat{G}\|_\infty = \sup_\omega |\hat{G}(j\omega)|.$$

It is left as an exercise to show that these are actually norms that satisfy the properties in Definition 1.2. The 2-norm is a measure of the energy of the impulse response of the system by making use of *Parseval's theorem*:

$$\|\hat{G}\|_2 = \int_{-\infty}^{\infty} |G(t)|^2\, dt.$$

The ∞-norm can be though of in multiple ways: it is the peak value of the frequency response of the system represented by $\hat{G}$ or, equivalently, the distance in the complex plane to the farthest point on the Nyquist plot of $\hat{G}$ (see FBS2e for the definition of the Nyquist plot). It can be shown that the ∞-norm is *submultiplicative*:

$$\|\hat{G}\hat{H}\|_\infty \leq \|\hat{G}\|_\infty \|\hat{H}\|_\infty.$$

For a linear, time-invariant (LTI) state space model of the form

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du,$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $y \in \mathbb{R}$, it can be shown that the transfer function has the form

$$\hat{G}(s) = C(sI - A)^{-1}B + D = \frac{n(s)}{d(s)}$$

where $n(s)$ and $d(s)$ are polynomials and $d(s)$ has highest order n. The *poles* of G are the roots of the denominator polynomial and the *zeros* of G are the roots of the numerator polynomial. We say that a transfer function $\hat{G}$ is *proper* if $\hat{G}(j\infty)$ is finite (in which case $\deg d \geq \deg n$), *strictly proper* if $\hat{G}(j\infty) = 0$ ($\deg d > \deg n$), and *biproper* if $\hat{G}$ and $\hat{G}^{-1}$ are both proper ($\deg d = \deg n$). The transfer function is said to be *stable* if it is analytic in the closed right half-plane (i.e., there are no right half-plane poles).

The following result is sometimes useful in proofs and derivations.

Theorem 1.1. *The 2-norm (respectively ∞-norm) of a rational transfer function $\hat{G}$ is finite if and only if $\hat{G}$ is strictly proper (respectively proper) and has no poles on the imaginary axis.*

1.4 System Norms

Given a norm for input signals and a norm for output signals, we can define the *induced norm* for an input/output system. Although this can be done for the general case of nonlinear input/output systems, we restrict ourselves here to the case of a linear input/output system. We furthermore assume that the input/output response is represented by the transfer function (hence we consider only the zero-state response).

Definition 1.7. The *induced a to b norm* for a linear system G is given by

$$\|G\|_{b,a} = \sup_{\|u\|_a \leq 1} \|y\|_b \qquad \text{where } y = G * u.$$

The induced a-norm to b-norm for a system is also called the *system gain.*

Theorem 1.2. *Assume that $\hat{G}$ is stable and strictly proper and that $\mathcal{U}, \mathcal{Y} = \mathcal{P}(-\infty, \infty)$. Then the following table summarizes the induced norms of G:*

	$\|u\|_2$	$\|u\|_\infty$
$\|y\|_2$	$\|\hat{G}\|_\infty$	∞
$\|y\|_\infty$	$\|\hat{G}\|_2$	$\|G\|_1$

Sketch of proofs.

2-norm to 2-norm. We first show that the 2-norm to 2-norm system gain is less than or equal to $\|\hat{G}\|_\infty$:

$$\begin{aligned}
\|y\|_2^2 &= \|\hat{Y}\|_2^2 \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 \cdot |\hat{U}(j\omega)|^2 \, d\omega \\
&\leq \|\hat{G}\|_\infty^2 \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{U}(j\omega)|^2 \, d\omega \\
&\leq \|\hat{G}\|_\infty^2 \cdot \|\hat{U}\|_2^2 = \|\hat{G}\|_\infty^2 \cdot \|u\|_2^2.
\end{aligned}$$

To establish equality it suffices to show that we can find an input that achieves the bound.

Let ω_0 be a frequency such that $\|\hat{G}(j\omega_0)\| = \|\hat{G}\|_\infty$ (this exists because $\hat{G}$ is stable and strictly proper). Define a signal u_ϵ such that

$$|\hat{U}_\epsilon(j\omega)| = \begin{cases} \sqrt{\pi/3} & \text{if } \omega_0 - \epsilon \leq \omega \leq \omega_0 + \epsilon \\ 0 & \text{otherwise} \end{cases}$$

and hence $\|u_\epsilon\|_2 = 1$. Then

$$\begin{aligned}
\|\hat{Y}_\epsilon\|_2^2 &= \frac{1}{2\pi} \int_{\omega_0-\epsilon}^{\omega_0+\epsilon} |\hat{G}(j\omega)|^2 \left(\frac{\pi}{\epsilon}\right) \, d\omega \\
&= \frac{1}{2\pi} \int_{\omega_0-\epsilon}^{\omega_0+\epsilon} |\hat{G}(j\omega_0)|^2 \left(\frac{\pi}{\epsilon}\right) \, d\omega + \delta_\epsilon \\
&= \|\hat{G}\|_\infty^2 \|u_\epsilon\|_2^2 + \delta_\epsilon,
\end{aligned}$$

where δ_ϵ represents the error that we obtain by evaluating $\hat{G}$ at $s = j\omega_0$ instead of $s = j\omega$ in the integral. By definition $\delta_\epsilon \to 0$ as $\epsilon \to 0$ (since $\hat{G}$ is continuous) and hence

$$\|\hat{Y}_0\|_2^2 = \|\hat{G}\|_\infty^2 \|\hat{U}_0\|^2$$

and so this input achieves the bound. (Note: to be more formal we need to rely on the fact that $\mathcal{U}$ and $\mathcal{Y}$ are Banach spaces.

∞-norm to 2-norm. Consider the bounded input $u(t) = 1$. This gives a constant output $y(t) = G(0)u(t)$. Assuming that the system has non-zero gain at $\omega = 0$ then $\|y\|_2 = \infty$. (If the gain is zero at zero frequency, a similar argument is possible using a sinusoid $u = \sin(\omega t)$.)

2-norm to ∞-norm. We make use of the following corollary of the Cauchy-Schwartz inequality:

$$\left(\int_{t_0}^{t_1} u(t)v(t)\, dt\right)^2 \leq \left(\int_{t_0}^{t_1} |u(t)|^2\, dt\right)\left(\int_{t_0}^{t_1} |v(t)|^2\, dt\right).$$

The output satisfies

$$\begin{aligned}|y(t)|^2 &= \left(\int_{-\infty}^{\infty} G(t-\tau)u(\tau)\, d\tau\right)^2\\ &\leq \left(\int_{-\infty}^{\infty} |G(t-\tau)|^2\, d\tau\right) \cdot \left(\int_{-\infty}^{\infty} |u(\tau)|^2\, d\tau\right)\\ &= \|G\|_2^2\, \|u\|_2^2 = \|\hat{G}\|_2^2\, \|u\|_2^2.\end{aligned}$$

Since this holds for all t, it follows that

$$\|y\|_\infty \leq \|\hat{G}\|_2 \|u\|_2.$$

To get equality, we can apply the signal $u(t) = G(-t)/\|G\|_2$. We have the $\|u\|_2 = 1$ and

$$|y(0)| = \int_{-\infty}^{\infty} G(-t)G(-t)/\|G\|_2\, dt = \|G\|_2.$$

So $\|y\|_\infty \geq |y(0)| = \|\hat{G}\|_2\|u\|_2$. Combining the two inequalities we have that $\|y\|_\infty = \|\hat{G}\|_2\|u\|_2$.

∞-norm to ∞-norm. See DFT [4].

□

1.5 Exercises

1.1 (DFT 2.1) Suppose that $u(t)$ is a continuous signal whose derivative $\dot{u}(t)$ is also continuous. Which of the following quantities qualifies as a norm for u:

(a) $\sup_t |\dot{u}(t)|$

(b) $|u(0)| + \sup_t |\dot{u}(t)|$

(c) $\max\{\sup_t |u(t)|, \sup_t |\dot{u}(t)|\}$

(d) $\sup_t |u(t)| + \sup_t |\dot{u}(t)|$

Make sure to give a thorough answer (not just yes or no).

1.2 (DFT 2.2) Consider the Venn diagram in Figure 2.1 of DFT. Show that the functions u1 to u9, defined below, are located in the diagram as shown in Figure 2.2. All the functions are zero for $t < 0$xs.

$$u_1(t) = \begin{cases} 1/\sqrt{t}, & \text{if } t \leq 1 \\ 0, & \text{if } t > 1 \end{cases}$$

$$u_2(t) = \begin{cases} 1/t^{\frac{1}{4}}, & \text{if } t \leq 1 \\ 0, & \text{if } t > 1 \end{cases}$$

$$u_3(t) = 1$$

$$u_4(t) = 1/(1+t)$$

$$u_5(t) = u_2 + u_4$$

$$u_6(t) = 0$$

$$u_7(t) = u_2(t) + 1$$

1.3 (DFT 2.4) Let D be a pure time delay of τ seconds with transfer function

$$\widehat{D}(s) = e^{-s\tau}.$$

A norm $\|\cdot\|$ on transfer functions is *time-delay invariant* if for every bounded transfer function $\widehat{G}$ and every $\tau > 0$ we have

$$\|\widehat{D}\widehat{G}\| = \|\widehat{G}\|$$

Determine if the 2-norm and ∞-norm are time-delay invariant.

1.4 Consider a discrete time system having dynamics

$$x[k+1] = Ax[k] + Bu[k], \qquad y[k] = Cx[k],$$

where $x[k] \in \mathbb{R}^n$ is the state of the system at time $k \in \mathbb{Z}$, $u[k] \in \mathbb{R}$ is the (scalar) input for the system, $y[k] \in \mathbb{R}$ is the (scalar) output for the system and A, B, and C are constant matrices of the appropriate size. We use the notation $x[k] = x(kh)$ to represent the state of the system at discrete time k where $h \in \mathbb{R}$ is the sampling time (and similarly for $u[k]$ and $y[k]$).

Let $\mathcal{T} = [0, h, \ldots, Nh]$ represent a discrete time range, with $N \in \mathbb{Z}$.

(a) Considered as a dynamical system over $\mathcal{T}$, what is the input space $\mathcal{U}$, output space $\mathcal{Y}$, and state space Σ corresponding to the dynamics above? Show that each of these spaces is a linear space by verifying the required properties (you may assume that $\mathbb{R}^p$ is a linear space for appropriate p).

(b) What is the state transition function $s(t_1, t_0, x_0, u(\cdot))$? Show that this function satisfies the state transition axiom and the semi-group axiom.

(c) What is the readout function $r(t, x, u)$? Show that the input/output system is a linear input/output dynamical system over $\mathcal{T}$.

(d) What is the zero-input response for the system? What is the zero-state response for the system?

1.5 (DFT 2.5) Compute the 1-norm of the impulse response corresponding to the transfer function

$$\frac{1}{\tau s + 1} \qquad \tau > 0.$$

1.6 (DFT 2.6) For $\hat{G}$ stable and strictly proper, show that $\|G\|_1 < \infty$ and find an inequality relating $\|\hat{G}\|_\infty$ and $\|G\|_1$. (Remember that G represents the impulse response corresponding to the transfer function $\hat{G}$.)

1.7 (DFT 2.7) Derive the ∞-norm to ∞-norm system gain for a stable, proper plant $\widehat{G}$. (Hint: write $\widehat{G} = c + \widehat{G}_1$ where c is a constant and $\widehat{G}_1$ is strictly proper.)

1.8 (DFT 2.8) Let $\widehat{G}$ be the transfer function for a stable, proper plant (but not necessarily strictly proper).

(a) Show that the ∞-norm of the output y given an input $u(t) = \sin(\omega t)$ is $|\widehat{G}(jw)|$.

(b) Show that the 2-norm to 2-norm system gain for $\widehat{G}$ is $\|\widehat{G}\|_\infty$ (just as in the strictly proper case).

1.9 (DFT 2.10) Consider a system with transfer function

$$\widehat{G}(s) = \frac{s+2}{4s+1}$$

and input u and output y. Compute

$$\|G\|_1 = \sup_{\|u\|_\infty = 1} \|y\|_\infty$$

and find an input that achieves the supremum.

1.10 (DFT 2.12) For a linear system with input u and output y, prove that

$$\sup_{\|u\| \le 1} \|y\| = \sup_{\|u\| = 1} \|y\|$$

where $\|\cdot\|$ is any norm on signals.

1.11 Consider a second order mechanical system with transfer function

$$\widehat{G}(s) = \frac{1}{s^2 + 2\omega_n \zeta s + \omega_n^2}$$

(ω_n is the natural frequency of the system and ζ is the damping ratio). Setting $\omega_n = 1$, plot the ∞-norm as a function of the damping ratio $\zeta > 0$. (You may use a computer to to this, but if you do then make sure to turn in a copy of your code with your solutions.)

Chapter 2

Linear Input/Output Systems

2.1 Matrix Exponential

Let $x(t) \in \mathbb{R}^n$ represent that state of a system whose dynamics satisfy the linear differential equation

$$\frac{d}{dt}x(t) = Ax(t), \qquad A \in \mathbb{R}^{n\times n},\, t \in [0, \infty).$$

The *initial value problem* is to find $x(t)$ given $x(0)$. The approach that we take is to show that there is a unique solution of the form $x(t) = e^{At}x(0)$ and then determine the properties of the solution (e.g., stability) as a function of the properties of the matrix A.

Definition 2.1. Let $S \in \mathbb{R}^{n\times n}$ be a square matrix. The *matrix exponential* of S is given by

$$e^S = I + S + \frac{1}{2}S^2 + \frac{1}{3!}S^2 + \dots + \frac{1}{k!}S^k + \dots$$

Proposition 2.1. *The series* $\sum_{k=0}^{\infty} \frac{1}{k!}S^k$ *converges for all* $S \in \mathbb{R}^{n\times n}$.

Proof. Simple case: Suppose S has a basis of eigenvectors $\{v_1, \dots, v_n\}$. Then

$$\begin{aligned} e^S v_i &= (I + S + \dots + \frac{1}{k!}S^k + \dots)v_i \\ &= (1 + \lambda_i + \dots + \frac{1}{k!}\lambda_i^k + \dots)v_i \\ &= e^{\lambda_i}v_i, \end{aligned}$$

which implies that $e^S x$ is well defined and finite (since this is true for all basis elements.

General case: Let $\|S\| = a$. Then

$$\|\frac{1}{k!}S^k\| \le \frac{1}{k!}\|S\|^k = \frac{a^k}{k!}.$$

Hence

$$\|e^S\| \le \sum_{k=1}^{\infty} \frac{a^k}{k!} = e^a = e^{\|S\|}$$

and so $e^S x$ is well-defined and finite. □

Proposition 2.2. *If $P, T \in \mathbb{R}^n \to \mathbb{R}^n$ and $S = PTP^{-1}$ then*

$$e^S = e^{PTP^{-1}} = Pe^T P^{-1}.$$

Proof.

$$\begin{aligned} e^S &= \sum \frac{1}{k!} S^k = \sum \frac{1}{k!} (PTP^{-1})^k \\ &= \dots \frac{1}{k!} (PTP^{-1}) \cdot (PTP^{-1}) \cdots (PTP^{-1}) \cdots \\ &= \sum \frac{1}{k!} PT^k P^{-1}) = P\Big(\sum \frac{T^k}{k!}\Big) P^{-1}) = Pe^T P^{-1}. \end{aligned}$$

□

Proposition 2.3. *If $S, T : \mathbb{R}^n \to \mathbb{R}^n$ commute ($ST = TS$) then $e^{S+T} = e^S e^T$.*

Proof. (basic idea) The first few terms of expansion for the matrix exponential are given by

$$\begin{aligned} (S+T)^0 &= I \\ (S+T)^1 &= S + T \\ (S+T)^2 &= (S+T)(S+T) = S^2 + ST + TS + T^2 \\ &= S^2 + 2ST + T^2 \quad \text{only if ST = TS!} \\ (S+T)^0 &= (S+T)(S+T)(S+T) \\ &= S^3 + S^2T + STS + ST^2 + TS^2 + TST + T^2S + T^3 \\ &= S^3 + 3S^2T + 3ST^2 + T^2 \quad \text{only if ST = TS!.} \end{aligned}$$

The general form becomes

$$(S+T)^k = \underbrace{\sum_{i=1}^{k} \binom{k}{i} S^i T^{k-i}}_{\text{binomial theorem}} = \sum_{i=1}^{k} \frac{k!}{i!(k-i)!} S^i T^{k-i}.$$

□

2.2 Convolution Equation

We now extend our results to include an input. Consider the non-autonomous differential equation

$$\dot{x} = Ax + b(t), \qquad x(0) = x_0. \tag{S2.1}$$

Theorem 2.4. *If $b(t)$ is a (piecewise) continuous signal, then there is a unique $x(t)$ satisfying equation* (S2.1) *given by*

$$x(t) = e^{At}x_0 + \int_0^t e^{T(t-\tau)} b(\tau)\, d\tau.$$

Proof. (existence only) Note that $x(0) = x_0$ and

$$\begin{aligned}
\frac{d}{dt}x(t) &= Ax(t) + \frac{d}{dt}\left(e^{At}\int_0^t e^{-A\tau}b(\tau)\,d\tau\right) \\
&= Ax(t) = Ae^{At}\left(\int_0^t e^{-A\tau}b(\tau)\,d\tau\right) + e^{At}\left(e^{-At}b(t)\right) \\
&= \left[Ax(t) + A\int_0^t e^{A(t-\tau)}b(\tau)\,d\tau\right] + b(t) \\
&= Ax(t) + b(t).
\end{aligned}$$

□

Note that the form of the solution is a combination of the initial condition response ($e^{At}x_0$) and the forced response ($\int_0^t \dots$). Linearity in the initial condition and the input follows from linearity of matrix multiplication and integration.

An alternative form of the solution can be obtained by defining the *fundamental matrix* $\Phi(t) = e^{At}$ as the solution of the matrix differential equation

$$\dot{\Phi} = A\Phi, \quad \Phi(0) = I.$$

Then the solution can be written as

$$x(t) = \Phi(t)x_0 + \underbrace{\int_0^t \Phi(t-\tau)b(\tau)\,d\tau}_{\textit{convolution}\text{ of }\Phi\text{ and }b(t)}.$$

Φ thus acts as a Green's function.

A common situation is that $b(t) = B \cdot a\sin(\omega t)$ where $B \in \mathbb{R}^n$ is a vector and $a\sin(\omega t)$ is a sinusoid with amplitude a and frequency ω. In addition, we wish to consider a specific combination of states $y = Cx$, where $C : \mathbb{R}^n \to \mathbb{R}$:

$$\begin{aligned}
\dot{x}(t) &= Ax + Bu(t) & u(t) &= a\sin(\omega t) \\
y(t) &= Cx & x(0) &= x_0.
\end{aligned} \tag{S2.2}$$

Theorem 2.5. *Let $H(s) = C(sI - A)^{-1}B$ and define $M = |H(i\omega)|$, $\phi = \arg H(i\omega)$. Then the sinusoidal response for the system in equation* (S2.2) *is given by*

$$y(t) = Ce^{At}x(0) + aM\sin(\omega t + \phi).$$

A proof can be found in FBS or worked out by using $\sin(\omega t) = \frac{1}{2}(e^{i\omega t} - e^{-i\omega t})$. The function $H(i\omega)$ gives the *frequency response* for the linear system. The function $H : \mathbb{C} \to \mathbb{C}$ is called the *transfer function* for the system.

2.3 Linear System Subspaces

To study the properties of a linear dynamical system, we study the properties of the eigenvalues and eigenvectors of the dynamics matrix $A \in \mathbb{R}^{n\times n}$. We will make use of the *Jordan canonical form* for a matrix. Recall that given any matrix $A \in \mathbb{R}^{n\times n}$ there exists a transformation $T \in \mathbb{C}^{n\times n}$ such that

$$J = TA^{-1}T = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_N \end{bmatrix}, \qquad J_k = \begin{bmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{bmatrix} \in \mathbb{R}^{m_k\times m_k}.$$

This is the complex version of the Jordan form. There is also a real version with $T \in \mathbb{R}^{n\times n}$ in which case the Jordan blocks representing complex eigenvalues have the form

$$J_k = \left[\begin{array}{c|c|c|c} \begin{matrix} a_k & -b_k \\ b_k & a_k \end{matrix} & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 & 0 \\ \hline 0 & \ddots & \ddots & 0 \\ \hline 0 & 0 & \ddots & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \\ \hline 0 & 0 & 0 & \begin{matrix} a_k & -b_k \\ b_k & a_k \end{matrix} \end{array}\right]$$

In both the real and complex cases the transformation matrices T consist of a set of generalized eigenvectors $w_{k_1}, \ldots, w_{k_{m_k}}$ corresponding to the eigenvalue λ_k.

Returning now to the dynamics of a linear system, let $A \in \mathbb{R}^{n\times n}$ be a square matrix representing the dynamics matrix with eigenvalues $\lambda_j = a_j + ib_j$ and corresponding (generalized) eigenvectors $w_j = u_j + iv_j$ (with $v_j = 0$ if $b_j = 0$). Let $\mathcal{B}$ be a basis of $\mathbb{R}^n$ given by

$$\mathcal{B} = \{\underbrace{u_1, \ldots, u_p,}_{\text{real } \lambda_j} \underbrace{u_{p+1}, v_{p+1}, \ldots, u_{p+q}, v_{p+q}}_{\text{complex } \lambda_j}\}. \tag{S2.3}$$

Definition 2.2. Given $A \in \mathbb{R}^n$ and basis vector $\mathcal{B}$ as in equation (S2.3), define

1. *Stable subspace*: $E^s = \operatorname{span}\{u_j, v_j : a_j < 0\}$;
2. *Unstable subspace*: $E^u = \operatorname{span}\{u_j, v_j : a_j > 0\}$;
3. *Center subspace*: $E^c = \operatorname{span}\{u_j, v_j : a_j = 0\}$.

These three subspaces can be used to characterize the behavior an unforced linear system. Since $E^s \cap E^u = \{0\}$, $E^s \cap E^c = \{0\}$, and $E^c \cap E^u = \{0\}$, it follows that any vector x can be written as a unique decomposition

$$x = u + v + w, \qquad u \in E^s,\ v \in E^c,\ w \in E^u,$$

and thus $\mathbb{R}^n = E^s \oplus E^c \oplus E^u$ where $\oplus$ is the direct sum of two linear subspaces, defined as $S_1 \oplus S_2 = \{u + v : u \in S_1,\ v \in S_2\}$. If all eigenvalues of A have nonzero real part, so that $E^c = \{0\}$ then the linear system $\dot{x} = Ax$ is said to be *hyperbolic*.

Definition 2.3. A subspace $E \subset \mathbb{R}^n$ is *invariant* with represent to the matrix $A \in \mathbb{R}^{n\times n}$ if $AE \subset E$ and is *invariant with respect to the flow* $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ if $e^{At}E \subset E$ for all t.

Proposition 2.6. *Let E be the generalized eigenspace of A corresponding to an eigenvalue λ. Then $AE \subset E$.*

Proof. Let $\{v_1, \dots, v_k\}$ be a basis for the generalized eigenspace of A. Then for every $v \in E$ we can write

$$v = \sum \alpha_j v_j \qquad \Longrightarrow \qquad Av = \sum \alpha_j A v_j$$

where α_j is the eigenvalue for the generalized eigenspace. Since $v_1, \dots, v_k$ space the generalized eigenvectors, we know that $(\lambda I - A)_j^k v_j = 0$ for some minimal k_j associated with v_j. It follows that

$$(A - \lambda I)v_j \in \ker(A - \lambda I)^{k_j - 1} \subset E$$

and hence $Av_j = w + \alpha_j v_j$ where $w \in E$, which implies that $Av_j \in E$. □

Proposition 2.7. *The subspaces E^s, E^c, and E^u are all invariant under A and e^{At}.*

Proof. Invariance under A follows from Proposition 2.6. To show invariance of the flow note that

$$e^{At} = (I + At + \frac{1}{2}A^2t^2 + \dots)$$

so

$$e^{At}E \subset E \oplus AE \oplus A^2E \oplus \dots \subset E.$$

□

Theorem 2.8 (Stability of linear systems)**.** *The following statements are equivalent*

1. *$E^s = \mathbb{R}^n$ (i.e., all eigenvalues have negative real part);*
2. *For all $x_0 \in \mathbb{R}^n$, $\lim_{t\to\infty} e^{At}x_0 = 0$ (trajectories converge to the origin);*
3. *There exist constants $a, c, m, M > 0$ such that*

$$me^{-at}\|x_0\| \le \|e^{At}x_0\| \le Me^{-ct}\|x_0\|$$

(exponential rate of convergence).

Proof. To show the equivalence of (1) and (2) we assume without loss of generality that the matrix is transformed into (real) Jordan canonical form. It can be shown that each Jordan block J_k can be decomposed into a diagonal matrix $S_k = \lambda_k I$ and a nilpotent matrix N_k consisting of 1's on the superdiagonal. The properties of the decomposition $J_k = S_k + N_k$ are that S_k and N_k commute and $N_k^{m_k} = 0$ (so that N_k is *nilpotent*). From these two properties we have that

$$e^{J_k t} = e^{\lambda_k I t}e^{N_k t} = e^{\lambda_k t}(I + N_k + \frac{1}{2}N^2t^2 + \dots + \frac{1}{(m_k - 1)!}N^{m_k - 1}t^{m_k - 1}).$$

A similar decomposition is possible for complex eigenvalues, with the diagonal elements of $e^{S_k t}$ taking the form

$$e^{a_k t}\begin{bmatrix} \cos(b_k t) & -\sin(b_k t) \\ \sin(b_k t) & \cos(b_k t) \end{bmatrix}$$

and the matrix N_k being a block matrix with superdiagonal elements given by the 2×2 identity matrix.

For the real blocks we have $\lambda_k < 0$ and for the complex blocks we have the $a_k < 0$ and it follows that $e^{Jt} \to 0$ as $t \to \infty$ (making use of the fact that $e^{\lambda t} t^m \to 0$ for any $\lambda > 0$ and $m \geq 0$). It follows that (1) and (2) are thus equivalent.

To show (3) we need two additional facts, which we state without proof.

Lemma 2.9. *Let $T \in \mathbb{R}^{n \times n}$ be an invertible transformation and let $y = Tx$. Then there exists constants m and M such that*

$$m\|x\| \leq \|y\| \leq M\|x\|.$$

Lemma 2.10. *Let $A \in \mathbb{R}^{n \times n}$ and assume $\alpha < |\mathrm{Re}\,(\lambda)| < \beta$ for all eigenvalues λ. Then there exists a set of coordinates $y = Tx$ such that*

$$\alpha\|y\|^2 \leq y^T(TAT^{-1})y \leq \beta\|y\|^2.$$

Using these two lemmas we can account for the transformation in converting the system into Jordan canonical form. The only remaining element to prove is that a function of the form $h(t) = e^{\lambda_k t} t^m < \gamma e^{\lambda t}$ for some λ and $\gamma > 0$. This follows from the fact that a function of the form $e^{-\epsilon t} t^m l$ is continuous and zero at $t = 0$ and at $t = \infty$ and thus $e^{-\epsilon t} t^m l$ is bounded above and below. From this we can show (with a bit more work) that for any Jordan block J_k there exists $\gamma > 0$ and $\lambda_k < \lambda < 0$ such that $me^{-at} < \|e^{J_k t} x_0\| < Me^{-ct}$ where $a < \lambda_k < c < 0$. The full result follows by combining all of the various bounds. □

A number of other stability results can be derived along the same lines as the arguments above. For example, if $\mathbb{R}^n = E^u$ (all eigenvalues have positive real part) then all solutions to the initial value problem diverge, exponentially fast. If $\mathbb{R}^n = E^u \oplus E^s$ then we have a mixture of stable and unstable spaces. Any initial condition with a component in E^u diverges, but if $x_0 \in E^s$ then the solution converges to zero.

The unresolved case is when $E^c \neq \{0\}$. In this case, the solutions corresponding to this subspace will have the form

$$(I + Nt + \frac{1}{2}N^2t^2 + \cdots + \frac{1}{k!}N^k t^k$$

for real eigenvalues and

$$\begin{bmatrix} \cos(b_k t) & -\sin(b_k t) \\ \sin(b_k t) & \cos(b_k t) \end{bmatrix} (I + Nt + \frac{1}{2}N^2t^2 + \cdots + \frac{1}{k!}N^k t^k$$

for complex eigenvalues. Convergence in this subspace depends on N. If $N = 0$ then the solutions remain bounded but do not converge to the original (stable in the sense of Lyapunov). If $N \neq 0$ the solutions diverge, but closer than the exponential case. The case of a nonlinear system whose linearization as a non-trivial center subspace leads to a center "manifold" for the nonlinear system and stability depends on the nonlinear characteristics of the system.

2.4 Input/output stability

A system is called bounded input/bounded output (BIBO) stable if a bounded input gives a bounded output for all initial states. A system is called input to state stable (ISS) if $\|x(t)\| \leq \beta(\|x(0)\|) + \gamma(\|u\|)$ where β and γ are monotonically increasing functions that vanish at the origin.

2.5 Time-Varying Systems

Suppose that we have a time-varying ("non-autonomous"), nonhomogeneous linear system with dynamics of the form

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + b(t), \\ y &= C(t)x + d(t), \end{aligned} \qquad x(0) = x_0. \tag{S2.4}$$

A matrix $\Phi(t,s) \in \mathbb{R}^{n\times n}$ is called the *fundamental matrix* for $\dot{x} = A(t)x$ if

1. $\frac{d}{dt}\Phi(t,s) = A(t)\Phi(t,s)$ for all s;
2. $\Phi(s,s) = I$;
3. $\det \Phi(t,s) \neq$ for all s,t.

Proposition 2.11. *If $\Phi(t,s)$ exists for $\dot{x} = A(t)x$ then the solution to equation* (S2.4) *is given by*

$$x(t) = \Phi(t,0)x_0 + \int_0^t \Phi(t,\tau)b(\tau)\,d\tau.$$

This solution generalizes the solution for linear time-invariant systems and we see that the structure of the solution—an initial condition response combined with a convolution integral—is preserved. If $A(t) = A$ is a constant, then $\Phi(t,s) = e^{A(t-s)}$ and we recover our previous solution. The matrix $\Phi(t,s)$ is also called the *state transition matrix* since $x(t) = \Phi(t,s)x(s)$ and $\Phi(t,\tau)\Phi(\tau,s) = \Phi(t,s)$ for all $t > \tau > s$. Solutions for $\Phi(t,s)$ exists for many different time-varying systems, including periodic systems, systems that are sufficiently smooth and bounded, etc.

Example 2.1. Let

$$A(t) = \begin{bmatrix} -1 & e^{at} \\ 0 & -1 \end{bmatrix}.$$

To find $\Phi(t,s)$ we have to solve the matrix differential equation

$$\begin{bmatrix} \dot{\Phi}_{11} & \dot{\Phi}_{12} \\ \dot{\phi}_{21} & \dot{\Phi}_{22} \end{bmatrix} = \begin{bmatrix} -1 & e^{at} \\ 0^{-}1 & \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \phi_{21} & \Phi_{22} \end{bmatrix},$$

with $\Phi(s,s) = I$, which serves as an initial condition for the system. We can break the matrix equation into its individual elements. Beginning with the equations for the bottom row of the matrix, we have

$$\begin{aligned} \dot{\Phi}_{21} &= -\Phi_{21} &&\Longrightarrow &&\Phi_{21}(t,s) = e^{-(t-s)}\Phi_{21}(s,s) = 0, \\ \dot{\Phi}_{22} &= -\Phi_{22} &&\Longrightarrow &&\Phi_{22}(t,s) = e^{-(t-s)}\Phi_{22}(s,s) = e^{-(t-s)}. \end{aligned}$$

Now use Φ_{21} and Φ_{22} to solve for Φ_{11} and Φ_{12}:

$$\begin{aligned} &\dot{\Phi}_{11} = -\Phi_{11} \quad \Longrightarrow \quad \Phi_{11}(t,s) = e^{-(t-s)} \\ &\dot{\Phi}_{12} = -\Phi_{12} + e^{at}e^{-(t-s)} = -\Phi_{12}(t,s) = e^{(a-1)t+s}. \end{aligned}$$

This last equation is of the form $\dot{x} = -x + b(t)$ and so we can solve it using the solution for a linear differential equation:

$$
\begin{aligned}
\Phi_{12}(t,s) &= e^{-(t-s)}\Phi_{12}(s,s) + \int_s^t e^{-(t-\tau)}e^{(a-1)\tau+s}\,d\tau \\
&= e^{-(t-s)}\int_s^t e^{a\tau}\,d\tau = e^{-(t-s)}\left(\frac{1}{a}e^{a\tau}\right)\Big|_s^t \\
&= e^{-(t-s)}\left(\frac{1}{a}e^{at} - \frac{1}{a}e^{as}\right) \\
&= \frac{1}{a}e^{at-(t-s)} - \frac{1}{a}e^{as-(t-s)}.
\end{aligned}
$$

Combining all of the elements, the fundamental matrix is thus given by

$$
\Phi(t,s) = \begin{bmatrix} e^{-(t-s)} & \frac{1}{a}\left(e^{at-(t-s)} - e^{as-(t-s)}\right) \\ 0 & e^{-(t-s)} \end{bmatrix}.
$$

The properties of the fundamental matrix can be verified by direct calculation (and are left to the reader).

The solution for the unforced system ($b(t) = 0$) is given by

$$
x(t) = \Phi(t,0)x(0) = \begin{bmatrix} e^{-t} & \frac{1}{a}\left(e^{(a-1)t} - e^{-t}\right) \\ 0 & e^{-t} \end{bmatrix} x(0).
$$

We see that although the eigenvalues of $A(t)$ are both -1, if $a > 1$ then some solutions of the differential equation diverge. This is an example that illustrates that for a linear system $\dot{x} = A(t)x$ stability requires more than $\operatorname{Re}(\lambda_A) < 0$.

A common situation is one in which $A(t)$ is period with period T:

$$
\frac{dx}{dt} = A(t)x(t), \qquad A(t+T) = A(t).
$$

In this case, we can show that the fundamental matrix has the form

$$
\Phi(t+T,s) = \Phi(T,0)\Phi(t,s).
$$

This property allows us to compute the fundamental matrix just over the period $[0,T]$ (e.g., numerically) and use this to determine the fundamental matrix at any future time. Additional explotation of the structure of the problem is also possible, as the next theorem illustrates.

Theorem 2.12 (Floquet). *Let $A(t)$ be piecewise continuous and T-periodic. Define $P(t) \in \mathbb{R}^{n\times n}$ as*

$$
P(t) = \Phi(t,0)e^{-Bt}, \qquad B = \Phi(T,0).
$$

Then

1. *$P(t+T) = P(t)$;*
2. *$P(0) = I$ and $\det P(t) \neq 0$;*

3. $\Phi(t,s) = P(t)e^{B(t-s)}P^{-1}(s)$*;*

4. If we set $z(t) = P^{-1}(t)x(t)$ *then* $\dot{z} = Bz$*.*

A consequence of this theorem is that in “rotating” ccoordinates z we can determine the stability properties of the system by examination of the matrix B. In particular, if $z(t) \to 0$ then $x(t) \to 0$. For a proof, see Callier and Desoer [3].

2.6 Exercises

2.1 (FBS2e 6.1) Show that if $y(t)$ is the output of a linear time-invariant system corresponding to input $u(t)$, then the output corresponding to an input $\dot{u}(t)$ is given by $\dot{y}(t)$. (Hint: Use the definition of the derivative: $\dot{z}(t) = \lim_{\epsilon\to 0}\bigl(z(t+\epsilon) - z(t)\bigr)/\epsilon$.)

2.2 (FBS2e 6.2) Show that a signal $u(t)$ can be decomposed in terms of the impulse function $\delta(t)$ as

$$u(t) = \int_0^t \delta(t-\tau)u(\tau)\,d\tau$$

and use this decomposition plus the principle of superposition to show that the response of a linear, time-invariant system to an input $u(t)$ (assuming a zero initial condition) can be written as a convolution equation

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau,$$

where $h(t)$ is the impulse response of the system. (Hint: Use the definition of the Riemann integral.)

2.3 (FBS2e 6.4) Assume that $\zeta < 1$ and let $\omega_\mathrm{d} = \omega_0\sqrt{1-\zeta^2}$. Show that

$$\exp\begin{bmatrix} -\zeta\omega_0 & \omega_\mathrm{d} \\ -\omega_\mathrm{d} & -\zeta\omega_0 \end{bmatrix} t = e^{-\zeta\omega_0 t}\begin{bmatrix} \cos\omega_\mathrm{d} t & \sin\omega_\mathrm{d} t \\ -\sin\omega_\mathrm{d} t & \cos\omega_\mathrm{d} t \end{bmatrix}.$$

Also show that

$$\exp\left(\begin{bmatrix} -\omega_0 & \omega_0 \\ 0 & -\omega_0 \end{bmatrix} t\right) = e^{-\omega_0 t}\begin{bmatrix} 1 & \omega_0 t \\ 0 & 1 \end{bmatrix}.$$

Use the results of this problem and the convolution equation to compute the unit step response for a spring mass system

$$m\ddot{q} + c\dot{q} + kq = F$$

with initial condition $x(0)$.

2.4 (FBS2e 6.6) Consider a linear system with a Jordan form that is non-diagonal.

(a) Prove Proposition 6.3 in *Feedback Systems* by showing that if the system contains a real eigenvalue $\lambda = 0$ with a nontrivial Jordan block, then there exists an initial condition with a solution that grows in time.

(b) Extend this argument to the case of complex eigenvalues with $\operatorname{Re}\lambda = 0$ by using the block Jordan form

$$J_i = \begin{bmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{bmatrix}.$$

2.5 (FBS2e 6.8) Consider a linear discrete-time system of the form

$$x[k+1] = Ax[k] + Bu[k], \qquad y[k] \quad = Cx[k] + Du[k].$$

(a) Show that the general form of the output of a discrete-time linear system is given by the discrete-time convolution equation:

$$y[k] = CA^k x[0] + \sum_{j=0}^{k-1} CA^{k-j-1}Bu[j] + Du[k].$$

(b) Show that a discrete-time linear system is asymptotically stable if and only if all the eigenvalues of A have a magnitude strictly less than 1.

(c) Show that a discrete-time linear system is unstable if any of the eigenvalues of A have magnitude greater than 1.

(d) Derive conditions for stability of a discrete-time linear system having one or more eigenvalues with magnitude identically equal to 1. (Hint: use Jordan form.)

(e) Let $u[k] = \sin(\omega k)$ represent an oscillatory input with frequency $\omega < \pi$ (to avoid "aliasing"). Show that the steady-state component of the response has gain M and phase θ, where

$$Me^{i\theta} = C(e^{i\omega}I - A)^{-1}B + D.$$

(f) Show that if we have a nonlinear discrete-time system

$$\begin{aligned} x[k+1] &= f(x[k], u[k]), \qquad x[k] \in \mathbb{R}^n,\ u \in \mathbb{R}, \\ y[k] &= h(x[k], u[k]), \qquad y \in \mathbb{R}, \end{aligned}$$

then we can linearize the system around an equilibrium point $(x_\mathrm{e}, u_\mathrm{e})$ by defining the matrices A, B, C, and D as in equation (6.35).

2.6 Using the computation for the matrix exponential, show that equation (6.11) in *Feedback Systems* holds for the case of a 3×3 Jordan block. (Hint: Decompose the matrix into the form $S + N$, where S is a diagonal matrix.)

2.7 Consider a stable linear time-invariant system. Assume that the system is initially at rest and let the input be $u = \sin \omega t$, where ω is much larger than the magnitudes of the eigenvalues of the dynamics matrix. Show that the output is approximately given by

$$y(t) \approx |G(i\omega)| \sin\big(\omega t + \arg G(i\omega)\big) + \frac{1}{\omega}h(t),$$

where $G(s)$ is the frequency response of the system and $h(t)$ its impulse response.

2.8 Consider the system

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x,$$

which is stable but not asymptotically stable. Show that if the system is driven by the bounded input $u = \cos t$ then the output is unbounded.

Chapter 3

Reachability and Stabilization

Preliminary reading The material in this chapter extends the material in Chapter 7 in FBS2e. Readers should be familiar with the material in Sections 7.1 and 7.2 in preparation for the more advanced concepts discussed here.

3.1 Concepts and Definitions

Consider an input/output dynamical system $\mathcal{D} = (\mathcal{U}, \Sigma, \mathcal{Y}, s, r)$ as defined in Section 1.2.

Definition 3.1 (Reachability)**.** A state x_f is *reachable from* x_0 *in time* T if there exists an input $u : [0, T] \to \mathbb{R}^m$ such that $x_\mathrm{f} = s(T, t_0, x_0, u)$.

If x_f is reachable from x_0 in time T we will write

$$x_0 \underset{T}{\rightsquigarrow} x_\mathrm{f}$$

or sometimes just $x_0 \rightsquigarrow x_\mathrm{f}$ if there exists some T for which x_f is reachable from x_0 in time T. The set of all states that are reachable from x_0 in time less than or equal to T is written as

$$\mathcal{R}_{\leq T}(x_0) = \{x_\mathrm{f} \in \mathbb{R}^n : x_0 \underset{\tau}{\rightsquigarrow} x_\mathrm{f} \text{ for some } \tau \leq T\}.$$

Definition 3.2 (Reachable sytem)**.** An input/output dynamical system $\mathcal{D}$ is *reachable* if for every $x_0, x_\mathrm{f} \in \mathbb{R}^n$ there exists $T > 0$ such that $x_0 \underset{T}{\rightsquigarrow} x_\mathrm{f}$.

.

The notion of reachability captures the property that we can reach a any final point x_f starting from x_0 with some choice of input $u(\,\cdot\,)$. In many cases, it will be not be possible to reach *all* states x_f but it may be possible to reach an open neighborhood of such points.

Definition 3.3 (Small-time local controllability)**.** A system is *small-time locally controllable* (STLC) if for any $T > 0$ the set $\mathcal{R}_{\leq T}(x_0)$ contains a neighborhood of x_0.

The notions of reachability and (small-time local) controllability hold for arbitrary points in the state space, but we are often most interested in equilibrium points and our ability to stabilize a system via state feedback. To define this notion more precisely, we specialize to the case of a state space control systems whose dynamics can be written in the form

$$\frac{dx}{dt} = f(x, u), \qquad x(0) = x_0. \tag{S3.1}$$

Definition 3.4 (Stabilizability)**.** A control system with dynamics (S3.1) is *stabilizable* at an equilibrium x_{e} if there exists a control law $u = \alpha(x, x_{\mathrm{e}})$ such that

$$\frac{dx}{dt} = f(x, \alpha(x, x_{\mathrm{e}})) =: F(x)$$

is locally asymptotically stable at x_{e}.

The main distinction between reachability and stabilizability is that there may be regions of the state space that are not reachable via application of appropriate control inputs but the dynamics may be such that trajectories with initial conditions in those regions of the state space converge to the origin under the natural dynamics of the system. We will explore this concept more fully in the special case of linear time-invariant systems.

3.2 Reachability for Linear State Space Systems

Consider a linear, time-invariant system

$$\frac{dx}{dt} = Ax + Bu, \qquad x(0) = x_0, \tag{S3.2}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, and having state transition function. In this case the state transition function is given by the convolution equation,

$$x(t) = s(t, 0, x_0, u(\,\cdot\,)) = e^{At}Bx_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\, d\tau.$$

It can be shown that if a linear system is small-time locally controllable at the origin then it is small-time locally controllable at any point x_f, and furthermore that small-time local controllability is equivalent to reachability between any two points (Exercise 3.9).

The problem of reachability for a linear time-invariant system is the same as the general case: we wish to find an input $u(\,\cdot\,)$ that can steer the system from an initial condition x_0 to a final condition x_{f} in a given time T. Because the system is linear, without loss of generality we can take $x_0 = 0$ (if not, replace the final position x_{f} with $x_{\mathrm{f}} - e^{AT}x_0$. In addition, since the state space dynamics depend only on the matrices A and B, we will often state that the pair (A, B) is reachable, stabilizable, etc.

The simplest (and most commonly) used test for reachability for a linear system is to check that the reachability matrix is full rank:

$$\operatorname{rank} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n.$$

The rank test provides a simple method for checking for reachability, but has the disadvantage that doesn't provide any quantitative insight into how "hard" it might be to either reach a given state or to assign the eigenvalues of the closed loop systems.

A better method of characterizing the reachability properties of a linear system is to make use of the fact that the system defines a linear map between the input $u(\,\cdot\,) \in \mathcal{U}$ and the state $x(T) = x_f \in \Sigma$:

$$x(T) = \mathcal{L}_T u(\,\cdot\,) = \int_0^T e^{A(T-\tau)}Bu(\tau)\, d\tau. \tag{S3.3}$$

Recall that for a linear operator in finite dimensional spaces $L : \mathbb{R}^m \to \mathbb{R}^n$ with $m > n$ that the rank of the linear operator L is the same as the rank of the linear operator $LL^* : \mathbb{R}^n \to \mathbb{R}^n$ where $L^* : \mathbb{R}^n \to \mathbb{R}^m$ is the adjoint operator (given by L^T in the case of matrices). Furthermore, if L is surjective (onto) then the least squares inverse of L is given by

$$L^+ = L^*(LL^*)^{-1} \quad \text{and} \quad LL^+ = I \in \mathbb{R}^{n\times n}.$$

More generally, the adjoint operator can be defined on a linear map between Banach spaces by defining the dual of a Banach space V to be the space V^* of continuous linear functionals on V. Given a linear function $\omega \in V^*$ we write $\langle \omega, v\rangle := \omega(v)$ to represent the application of the function ω on the element v. In the case of finite dimensional vector spaces we can associate the set V^* with V and $\langle \sigma, v\rangle$ is of the form $w^\mathsf{T} v$ where $w \in \mathbb{R}^n$.

If we have a mapping between linear spaces V and W given by $L : V \to W$, the adjoint operator $L^* : W^* \to V^*$ is defined as the unique operator that satisfies

$$\langle L^*\sigma, v\rangle = \langle \sigma, Lv\rangle \quad \text{for all } v \in V \text{ and } \sigma \in W^*.$$

Note that the application of the linear function on the left occurs in the space V and on the right occurs in the space W.

For a signal space $\mathcal{U}$, a linear functional has the form of an integral

$$\langle \omega(\,\cdot\,), u(\,\cdot\,)\rangle = \int_0^\infty \omega(\tau)\cdot u(\tau)\, d\tau$$

and so we can associate each linear function in $\mathcal{U}^*$ with a function $\omega(t)$. Given a linear mapping $\mathcal{L}_T : \mathcal{U} \to \mathbb{R}^n$ of the form

$$\mathcal{L}_T(u(\,\cdot\,)) = \int_0^T h(T-\tau)u(\tau)\, d\tau$$

it can be shown that the adjoint operator $\mathcal{L}_T{}^* : \mathbb{R}^n \to \mathcal{U}^*$ is given by

$$\mathcal{L}_T^*(t)w^* = \begin{cases} \langle h(T-t), w\rangle & \text{if } t \leq T, \\ 0 & \text{otherwise} \end{cases}$$

where $w \in \mathbb{R}^n$.

To show that a system is reachable, we need to show that $\mathcal{L}_T : \mathcal{U} \to \mathbb{R}^n$ given by equation S3.3 is full rank. Using the analysis above, the adjoint operator $\mathcal{L}_T^* : \mathbb{R}^n \to \mathcal{U}^*$ is

$$(\mathcal{L}_T^* v)(t) = B^\mathsf{T} e^{A^\mathsf{T}(T-t)} v.$$

As in the finite dimensional case, the dimension of the range of the map $\mathcal{L}_T : \mathcal{U} \to \mathbb{R}^n$ is the same as the dimension of the range of the map $\mathcal{L}_T\mathcal{L}_T^* : \mathbb{R}^n \to \mathbb{R}^n$, which is given by

$$\mathcal{L}_T\mathcal{L}_T^* = \int_0^T e^{A(T-\tau)} BB^\mathsf{T} e^{A^\mathsf{T}(T-\tau)}\, d\tau.$$

This analysis leads to the following result on reachability for a linear system. △

Theorem 3.1 (Gramian test)**.** *A pair (A, B) is reachable in time T if and only if*

$$W_c(T) = \int_0^T e^{A(T-\tau)} B B^\mathsf{T} e^{A^\mathsf{T}(T-\tau)}\, d\tau = \int_0^T e^{A\tau} B B^\mathsf{T} e^{A^\mathsf{T}\tau}\, d\tau$$

is positive definite.

The matrix $W_\mathrm{c}(T)$ provides a means to compute an input that steers a linear system from the origin to a point $x_\mathrm{f} \in \mathbb{R}^n$. Given $T > 0$, define

$$u(t) = B^\mathsf{T} e^{A^\mathsf{T}(T-t)} W_\mathrm{c}^{-1}(T) x_\mathrm{f}.$$

It follows from the definition of W_c that $x_0 \underset{T}{\rightsquigarrow} x_f$. Furthermore, it is possible to show that if the system is reachable for some $T > 0$ then it is reachable for *all* $T > 0$. Note that this computation of $u(\,\cdot\,)$ corresponds to the computation of the least squares inverse in the finite dimensional case ($u = \mathcal{L}_T^*(\mathcal{L}_T \mathcal{L}_T)^{-1} x_\mathrm{f}$).

Lemma 3.2. *If $W_\mathrm{c}(T)$ is positive definite for some $T > 0$ then it is positive definite for all $T > 0$.*

Proof. We prove the statement by contradiction. Suppose that $W_\mathrm{c}(T)$ is positive definite for a specific $T > 0$ but that there exists $T' > 0$ such that $\operatorname{rank} W_\mathrm{c}(T') = k < n$. Then there exists a vector $v \in \mathbb{R}^n$ such that $v^\mathsf{T} W_\mathrm{c}(T') = 0$ and furthermore

$$v^\mathsf{T} W_\mathrm{c}(T') v = v^\mathsf{T} \left(\int_0^{T'} e^{A\tau} B B^\mathsf{T} e^{A^\mathsf{T}\tau} v \, d\tau \right) = 0.$$

Since the integrand is a symmetric matrix, it follows that we must have

$$v^\mathsf{T} e^{A\tau} B B^\mathsf{T} e^{A^\mathsf{T}\tau} v = 0 \quad \text{for all } \tau \le T',$$

and hence

$$\begin{aligned} v^\mathsf{T} e^{A\tau} B = 0 \quad &\Longrightarrow \quad v^\mathsf{T} B = 0 \quad \text{(evaluating at } t = 0\text{)} \\ \frac{d}{d\tau}(v^\mathsf{T} e^{A\tau} B) = v^\mathsf{T} A e^{A\tau} B = 0 \quad &\Longrightarrow \quad v^\mathsf{T} A B = 0 \\ &\vdots \\ &v^\mathsf{T} A^{n-1} B = 0. \end{aligned}$$

Therefore $v^\mathsf{T} e^{A\tau} B = 0$ for *all* τ (including $\tau > T'$) and hence $v^\mathsf{T} W_\mathrm{c}(t) = 0$ for all $t > 0$, contradicting our original hypothesis. □

If the eigenvalues of A all have negative real part, it can be shown that $W_c(t)$ converges to a constant matrix as $t \to \infty$ and we write this matrix as $W_c = W_c(\infty)$. This matrix is called the *controllability Gramian.* (Note that FBS2e uses W_r to represent the reachability *matrix* $[B\ AB\ A^2B\ \ldots]$. This is different than the controllability *Gramian.*)

Theorem 3.3. $AW_c + W_c A^\mathsf{T} = -BB^\mathsf{T}$.

Proof.

$$\begin{aligned}
AW_\mathrm{c} + W_\mathrm{c}A^\mathsf{T} &= \int_0^\infty Ae^{A\tau}BB^\mathsf{T}e^{A^\mathsf{T}\tau}\,d\tau + \int_0^\infty e^{A\tau}BB^\mathsf{T}e^{A^\mathsf{T}\tau}A^\mathsf{T}\,d\tau \\
&= \int_0^\infty \frac{d}{dt}\left(e^{A\tau}BB^\mathsf{T}e^{A^\mathsf{T}\tau}A^\mathsf{T}\right)\,d\tau \\
&= \left(e^{At}BB^\mathsf{T}e^{A^\mathsf{T}t}\right)\Big|_{t=0}^{\infty} \\
&= 0 - BB^\mathsf{T} = -BB^\mathsf{T}.
\end{aligned}$$

□

Theorem 3.4. *A linear time-invariant control system S3.2 is reachable if and only if W_c is full rank and the subspace of points that are reachable from the origin is given by the image of W_c.*

Proof. Left as an exercise. Use the fact that the range of $W_\mathrm{c}(T)$ is independent of T. □

Reachability is best captured by the Gramian since it relates directly to the map between an input vector and final state, and its norm is related to the difficulty of moving from the origin to an arbitrary state. Furthermore, the eigenvectors of W_c and the corresponding eigenvalues provide a measure of how much control effort is required to move in different directions. There are, however, several other tests for reachability that can be used for linear systems.

Theorem 3.5. *The following conditions are necessary and sufficient for reachability of a linear time-invariant system:*

- *Reachability matrix test:*

$$\mathit{rank}\begin{bmatrix}B & AB & \cdots & A^{n-1}B\end{bmatrix} = n.$$

- *Popov-Belman-Hautus (PBH) test:*

$$\mathit{rank}\left[\begin{array}{c|c} sI - A & B \end{array}\right] = n$$

  for all $s \in \mathbb{C}$ (suffices to check for eigenvalues of A).

Proof. (Incomplete) PBH necessity: Suppose

$$\mathrm{rank}\left[\begin{array}{c|c} \lambda I - A & B \end{array}\right] < n.$$

Then there exists $v \neq 0$ such that

$$v^\mathsf{T}\left[\begin{array}{c|c} \lambda I - A & B \end{array}\right] = 0$$

and hence $x^\mathsf{T}A = \lambda x^\mathsf{T}$ and $x^\mathsf{T}B = 0$. It follows that $x^\mathsf{T}A^2 = \lambda^2 x^\mathsf{T}, \ldots, x^\mathsf{T}A^{n-1} = \lambda^{n-1}x^\mathsf{T}$ and thus

$$x^\mathsf{T}\begin{bmatrix}B & AB & \cdots & A^{n-1}B\end{bmatrix} = 0.$$

□

For both of these tests, we note that if the corresponding matrix is rank deficient, the left null space of that matrix gives directions in the state space that are unreachable (more accurately it consists of the directions in which the projected value of the state is constant along all trajectories of the system). The set of vectors orthogonal to this left null space defines a subspace V_r that represents the set of reachable states (exercise: prove this is a subspace).

Theorem 3.6. *Assume (A, B) is not reachable. Let $\text{rank}\, W_c = r < n$. Then there exists a transformation $T \in \mathbb{R}^{n\times n}$ such that*

$$TAT^{-1} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \quad TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

where $A_1 \in \mathbb{R}^{r\times r}$, $B_1 \in \mathbb{R}^{r\times m}$, and (A_1, B_1) is reachable.

Proof. (Sketch) Let V_{r} represent the null space of W_{c} and let $\mathcal{B}_{\bar{\text{r}}} = \{w_1, \dots, w_{n-r}\}$ represent a basis for V_{r}. Complete this basis with a set of vectors $\{v_1, \dots, v_r\}$ such that $\{v_1, \dots, v_r, w_1, \dots, w_{n-r}\}$ is a basis for $\mathbb{R}^n$. Use these basis vectors as the columns of the transformation T. □

We note that the null space of W_{c} is uniquely defined, though the basis for that space is not unique. This subspace represents the set of linear functions on the state space whose values are constant and hence provides a characterization of the unreachable states of the system. The complement of that space is not a subspace, although if we look at the points that are reachable from the origin, this does form a subspace. We will return to this point in more detail when we discuss the Kalman decomposition in Chapter 5. △

Finally, we note that a system that is reachable can be written in *reachable canonical form* (see FBS2e). This is primarily useful for proofs.

3.3 System Norms

3.3.1 State space computation of the 2-norm

Consider a stable state space system with no direct term and with system matrices A, B, and C. Let W_{c} be the controllability Gramian for the system and let $G(t)$ represent the impulse response function for the system and $\hat{G}(s)$ represent the corresponding transfer function (Laplace transform of the impulse response). Recall that the 2-norm to ∞-norm gain for a linear input/output system is given by $\|\hat{G}\|_2$.

Theorem 3.7. $\|\hat{G}\|_2 = \sqrt{CW_cC^\mathsf{T}}$.

Proof. The impulse response function given by

$$\begin{aligned} G(t) = Cx_\delta(t) &= C\int_0^t A^{A(t-\tau)} B\delta(\tau)\, d\tau \\ &= Ce^{At}B, \qquad t > 0. \end{aligned}$$

The system norm is given by

$$\begin{aligned} \|\hat{G}\|_2^2 &= \|G\|_2^2 \\ &= \int_0^\infty \big(Ce^{A\tau}B\big)\big(B^\mathsf{T} e^{A^\mathsf{T}\tau} C^\mathsf{T}\big)\, d\tau \\ &= C\left(\int_0^\infty e^{At}BB^\mathsf{T} e^{A^\mathsf{T}\tau}\, d\tau\right) C^\mathsf{T} \\ &= CW_{\text{c}}C^\mathsf{T}. \end{aligned}$$

□

The more common norm in control system design is the 2-norm to 2-norm system gain, which is given by $\|\hat{G}\|_\infty$. To compute the ∞-norm of a transfer function, we define

$$H_\gamma = \begin{bmatrix} A & \frac{1}{\gamma}BB^\mathsf{T} \\ -C^\mathsf{T}C & -A^\mathsf{T} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}.$$

The system gain can be determined in terms of H_γ as follows.

Theorem 3.8. *$\|\hat{G}\|_\infty < \gamma$ is an only if H_γ has no eigenvalues on the $j\omega$ axis.*

Proof. DGKF. □

To numerically compute the H_∞ norm, we can use the bisection method to determine γ to arbitrary accuracy.

3.4 Stabilization via Linear Feedback

We now consider the problem of stabilization, as defined in Definition 3.4. For a linear system, we will consider feedback laws of the form $u = -Kx$ (the negative sign is a convention associated with the use of "negative" feedback), so that

$$\frac{dx}{dt} = Ax + Bu = (A - BK)x.$$

One of the goals of introducing negative feedback is to stabilize an otherwise unstable system at the origin. In addition, state feedback can be used to "design the dynamics" of the close loop system by attempting to assign the eigenvalues of the closed loop system to specific values.

Theorem 7.3 states that if a (single-input) system is reachable then it is possible to assign the eigenvalues of the closed loop system to arbitrary values. This turns out to be true for the multi-input case as well and is proved in a similar manner (by using an appropriate normal form).

Using the decomposition theorem 3.6 it is easy to see that the question of stabilizability for a linear system comes down to the question of whether the dynamics in the unreachable space ($\dot{z} = A_3 z$) are stable, since these eigenvalues cannot be changed through the use of state feedback.

Although eigenvalue placement provides an easy method for designing the dynamics of the closed loop system, it is rarely used directly since it does not provide any guidelines for trading off the size of the inputs required to stabilize the dynamics versus the properties of the closed loop response. This is explored in a bit more detail in FBS2e Section 14.6 (Robust Pole Placement).

3.5 Exercises

3.1 (Sontag 3.1.2/3.1.3) Prove the following statements:

(a) If $(x, \sigma) \rightsquigarrow (z, \tau)$ and $(z, \tau) \rightsquigarrow (y, \mu)$, then $(x, \sigma) \rightsquigarrow (y, \mu)$.

(b) If $(x, \sigma) \rightsquigarrow (y, \mu)$ and if $\sigma < \tau < \mu$, then there exists a $z \in \mathcal{X}$ such that $(x, \sigma) \rightsquigarrow (z, \tau)$ and $(z, \tau) \rightsquigarrow (y, \mu)$.

(c) If $x \underset{T}{\rightsquigarrow} y$ for some $T > 0$ and if $0 < t < T$, then there is some $z \in \mathcal{X}$ such that $x \underset{t}{\rightsquigarrow} z$ and $z \underset{T-t}{\rightsquigarrow} y$.

(d) If $x \underset{t}{\rightsquigarrow} z$, $z \underset{s}{\rightsquigarrow} y$, and Σ is time-invariant, then $x \underset{t+s}{\rightsquigarrow} y$.

(e) If $x \rightsquigarrow z$, $z \rightsquigarrow y$, and Σ is time-invariant, then $x \rightsquigarrow y$.

(f) Given examples that show that properties (d) and (e) may be false if Σ is not time-invariant.

(g) Even for time-invariant systems, it is not necessarily true that $x \rightsquigarrow z$ implies that $z \rightsquigarrow x$ (so, "$\rightsquigarrow$" is not an equivalence relation).

3.2 (FBS2e 7.1) Consider the double integrator. Find a piecewise constant control strategy that drives the system from the origin to the state $x = (1, 1)$.

3.3 (FBS2e 7.2) Extend the argument in Section 7.1 to show that if a system is reachable from an initial state of zero, it is reachable from a nonzero initial state.

3.4 (FBS2e 7.3) Consider a system with the state x and z described by the equations

$$\frac{dx}{dt} = Ax + Bu, \qquad \frac{dz}{dt} = Az + Bu.$$

If $x(0) = z(0)$ it follows that $x(t) = z(t)$ for all t regardless of the input that is applied. Show that this violates the definition of reachability and further show that the reachability matrix W_r is not full rank. What is the rank of the reachability matrix?

3.5 (FBS2e 7.6) Show that the characteristic polynomial for a system in reachable canonical form is given by equation (7.7) and that

$$\frac{d^n z_k}{dt^n} + a_1 \frac{d^{n-1} z_k}{dt^{n-1}} + \cdots + a_{n-1} \frac{dz_k}{dt} + a_n z_k = \frac{d^{n-k} u}{dt^{n-k}},$$

where z_k is the kth state.

3.6 (FBS2e 7.7) Consider a system in reachable canonical form. Show that the inverse of the reachability matrix is given by

$$\tilde{W}_\mathrm{r}^{-1} = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ & 1 & a_1 & \cdots & a_{n-2} \\ & & 1 & \ddots & \vdots \\ & 0 & & \ddots & a_1 \\ & & & & 1 \end{bmatrix}.$$

3.7 (FBS2e 7.10) Consider the system

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x,$$

with the control law

$$u = -k_1 x_1 - k_2 x_2 + k_\mathrm{f} r.$$

Compute the rank of the reachability matrix for the system and show that eigenvalues of the system cannot be assigned to arbitrary values.

3.8 (FBS2e 7.11) Let $A \in \mathbb{R}^{n\times n}$ be a matrix with characteristic polynomial $\lambda(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$. Show that the matrix A satisfies

$$\lambda(A) = A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I = 0,$$

where the zero on the right hand side represents a matrix of elements with all zeros. Use this result to show that A^n can be written in terms of lower order powers of A and hence any matrix polynomial in A can be rewritten using terms of order at most $n - 1$.

3.9 Show that for a linear time-invariant system, the following notions of controllability are equivalent:

(a) Reachability to the origin ($x_0 \rightsquigarrow 0$).

(b) Reachability from the origin ($0 \rightsquigarrow x_\mathrm{f}$).

(c) Small-time local controllability ($x_0 \rightsquigarrow B(x_0, \epsilon)$).

3.10 (Sontag 3.3.4) Assume that the pair (A, B) is not controllable with $\dim R(A, B) = r < n$. From Lemma 3.3.3, there exists a $T \in GL(n)$ such that the matrices $\tilde{A} := T^{-1}AT$ and $\tilde{B} := T^{-1}B$ have the block structure

$$\tilde{A} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

where $A_1 \in \mathbb{R}^{r\times r}$ and $B_1 \in \mathbb{R}^{r\times m}$. Prove that (A_1, B_1) is itself a controllable pair.

3.11 (Sontag 3.3.6) Prove that if

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

then (A, B) is controllable if and only if $\lambda_i \neq \lambda_j$ for each $i \neq j$ and all $b_i \neq 0$.

3.12 (Sontag 3.3.14) Let (A, B) correspond to a time-invariant *discrete-time* linear system Σ. Recall that null-controllability means that every state can be controlled to zero. Prove that the following conditions are equivalent:

(a) Σ is null-controllable.

(b) The image of A^n is contained in the image of $R(A, B)$.

(c) In the decomposition in Sontag, Lemma 3.3.3, A_3 is nilpotent.

(d) $\operatorname{rank}[\lambda I - A, B] = n$ for all nonzero $\lambda \in \tilde{\mathbb{R}}$.

(e) $\operatorname{rank}[\lambda I - A, B] = n$ for all $\lambda \in \tilde{\mathbb{R}}$.

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