CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences
CDS 131

R. Murray | **Homework Set #6** | Issued: 7 Nov 2018
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Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. [FBS 2.4] Consider a closed loop system with process dynamics and a PI controller modeled by
$$\frac{dy}{dt} + ay = bu, \qquad u = k_\text{p}(r - y) + k_\text{i} \int_0^t \big(r(\tau) - y(\tau)\big)\, d\tau,$$
where r is the reference, u is the control variable, and y is the process output.
   (a) Derive a differential equation relating the output y to the reference r by direct manipulation of the equations and compute the transfer function $H_{yr}(s)$.
   (b) Draw a block diagram of the system and derive the transfer functions of the process $P(s)$ and the controller $C(s)$.
   (c) Use block diagram algebra to compute the transfer function from reference r to output y of the closed loop system and verify that your answer matches your answer in part (a).

2. [FBS 9.8] Consider the linear state space system
$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx.$$
   (a) Show that the transfer function is
$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n},$$
   where
$$b_1 = CB, \quad b_2 = CAB + a_1 CB, \quad \dots, \quad b_n = CA^{n-1}B + a_1 CA^{n-2}B + \cdots + a_{n-1}CB$$
   and $\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_n$ is the characteristic polynomial for A.
   (b) Compute the transfer function for a linear system in reachable canonical form and show that it matches the transfer function given above.

3. [FBS 9.14] Consider the differential equation
$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_n y = b_1 \frac{d^{n-1}u}{dt^{n-1}} + b_2 \frac{d^{n-2}u}{dt^{n-2}} + \cdots + b_n u.$$
   (a) Let λ be a root of the characteristic polynomial
$$s^n + a_1 s^{n-1} + \cdots + a_n = 0.$$
   Show that if $u(t) = 0$, the differential equation has the solution $y(t) = e^{\lambda t}$.

(b) Let κ be a zero of the polynomial

$$b(s) = b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n.$$

Show that if the input is $u(t) = e^{\kappa t}$, then there is a solution to the differential equation that is identically zero.

4. [FBS 9.15] Consider a closed loop system of the form of Figure 9.6, with $F = 1$ and P and C having a pole/zero cancellation. Show that if each system is written in state space form, the resulting closed loop system is not reachable and not observable.

5. Consider a process P with dynamics

$$\frac{dx}{dt} = ku, \qquad y = x.$$

We wish to design a controller $C(s)$ using an observer-based optimal control law.

(a) Design a state feedback controller $u = -Kx$ for the system that minimizes the cost function

$$J = \int_0^\infty y^2 + u^2 \, dt$$

(assume that the full state is available for feedback).

(b) Design an observer for this system that places the closed loop pole for the observer at $s = -1$.

(c) Letting $\alpha = K$ represent the state feedback gain in part 5a and $\beta = L$ the observer feedback gain in part 5b, compute the controller transfer function resulting from applying the optimal state feedback gain to the estimated state. Under what conditions is the closed loop system stable?

6. [Dullerud and Paganini, 2.19] Consider a transfer function $G(s)$ and let (A, B, C, D) be a realization: $G(s) = C(sI - A)^{-1}B + D$.

(a) Prove that if (A, B, C, D) is a minimal realization for $G(s)$ then every eigenvalue of A must be a pole of $G(s)$.

(b) Prove that if (A, B, C, D) is a non-minimal realization, then every eigenvalue of A is either a pole of $G(s)$ or corresponds to an eigenvector that is in the unreachable or unobservable subspace.