CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 131

R. Murray | **Homework Set #3** | Issued: 17 Oct 2018

Fall 2018 | | Due: 24 Oct 2018

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. [Sontag 3.1.2/3.1.3] Prove the following statements:

   (a) If $(x, \sigma) \rightsquigarrow (z, \tau)$ and $(z, \tau) \rightsquigarrow (y, \mu)$, then $(x, \sigma) \rightsquigarrow (y, \mu)$.

   (b) If $(x, \sigma) \rightsquigarrow (y, \mu)$ and if $\sigma < \tau < \mu$, then there exists a $z \in \mathcal{X}$ such that $(x, \sigma) \rightsquigarrow (z, \tau)$ and $(z, \tau) \rightsquigarrow (y, \mu)$.

   (c) If $x \underset{t}{\rightsquigarrow} z$, $z \underset{s}{\rightsquigarrow} y$, and Σ is time-invariant, then $x \underset{t+s}{\rightsquigarrow} y$.

2. [FBS 7.1] Consider the double integrator. Find a piecewise constant control strategy that drives the system from the origin to the state $x = (1, 1)$.

3. [FBS 7.2] Extend the argument in Section 7.1 to show that if a system is reachable from an initial state of zero, it is reachable from a nonzero initial state.

4. [FBS 7.6] Show that the characteristic polynomial for a system in reachable canonical form is given by equation (7.7) and that

$$\frac{d^n z_k}{dt^n} + a_1 \frac{d^{n-1} z_k}{dt^{n-1}} + \cdots + a_{n-1} \frac{dz_k}{dt} + a_n z_k = \frac{d^{n-k} u}{dt^{n-k}},$$

   where z_k is the kth state.

5. [FBS 7.7] Consider a system in reachable canonical form. Show that the inverse of the reachability matrix is given by

$$\tilde{W}_r^{-1} = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ \vdots & 0 & 1 & \ddots & \vdots \\ 0 & \cdots & 0 & \ddots & a_1 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

6. [FBS 7.9] Consider the system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x,$$

   with the control law

$$u = -k_1 x_1 - k_2 x_2 + k_f r.$$

   Compute the rank of the reachability matrix for the system and show that eigenvalues of the system cannot be assigned to arbitrary values.

7. [Sontag 3.3.4] Assume that the pair (A, B) is not controllable with $\dim R(A, B) = r < n$. From Lemma 3.3.3, there exists a $T \in GL(n)$ such that the matrices $\tilde{A} := T^{-1}AT$ and $\tilde{B} := T^{-1}B$ have the block structure

$$\tilde{A} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

where $A_1 \in \mathbb{R}^{r\times r}$ and $B_1 \in \mathbb{R}^{r\times m}$. Prove that (A_1, B_1) is itself a controllable pair.

8. [Sontag 3.3.6] Prove that if

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix}$$

then (A, B) is controllable if and only if $\lambda_i \neq \lambda_j$ for each $i \neq j$ and all $b_i \neq 0$.