CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 112/Ae 103b

Homework Set #9

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Problem 1. The Lorenz system is a system of first order differential equations which has been widely studied due to its chaotic solutions (for certain parameter values and initial conditions). The first order ODE which describes its evolution is:

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= x(\rho - z) - y\\ \dot{z} &= xy - \beta z\end{aligned}$$

We will consider the parameter set $\sigma = 10$, $\beta = 8/3$, $\rho = 28$. In this exercise you are tasked with estimating the state of this system using the extended Kalman filter as well as the unscented Kalman filter. Two scenarios are considered.

(a) For some non-zero initial condition of your choice, simulate the state of the Lorenz system using the `lorenz_noisy` system with zero mean white noise disturbances with noise intensity $R_V = \text{diag}([1, 1, 1])$. Corrupt the state with measurement noise $R_W = \text{diag}([1, 1, 1])$ and then provide plots of the estimated states when using an EKF vs a UKF. Comment on the performance of the two estimators.

(b) Now simulate the system again without process disturbances and corrupt the state with the same measurement noise as in the previous part. Now construct a new system which you will call `lorenz_noisy_perturbed` in which the model parameters have been perturbed: $\sigma \leftarrow \sigma + \epsilon$, $\beta \leftarrow \beta + \epsilon$, $\rho \leftarrow \rho + \epsilon$. Now run the EKF and UKF using this perturbed model and provide plots of the estimated states. Consider three different values of $\epsilon \in 0.1, 0.5, 1.0$. Comment on how the two estimators compare over different values of ϵ.

(c) Repeat both of the above steps in the case when only x and z are being measured ($R_W = \text{diag}([1, 1])$ and $R_V = \text{diag}([1, 1, 1])$).

In each scenario you should simulate the appropriate system using `ct.input\_output\_response()`, corrupt the state with zero-mean white measurement noise $R_W = \text{diag}([1, 1, 1])$ and then estimate the system using the predictor corrector form of the EKF as well as the predictor corrector form of the UKF. Simulate the system using a sample time of $T_s = 0.0005$ for $T_f = 15$ seconds.

Note: The files `lorenz.py` and `ukf.py` are available for the course website.

Problem 2. Consider the discrete time linear system

$$X[k+1] = \begin{bmatrix}0.5 & 1 & 0\\ 0 & -0.8 & 1\\ 0 & 0 & 0.5\end{bmatrix} X[k] + \begin{bmatrix}0\\0\\1\end{bmatrix} u + \begin{bmatrix}0\\1\\0\end{bmatrix} V, \qquad Y[k] \quad = \sin(\begin{bmatrix}1 & 0 & 0\end{bmatrix} X) + W,$$

where V is a discrete time, white noise process with covariance 0.01 and W is a discrete time, white noise process with covariance 10^{-4}. Let $u[k] = \sin(2\pi k/5)$ and assume that $P[0] = \mathbb{E}(X[0]X^\mathsf{T}[0]) = 0.5I$.

(a) Construct an optimal estimator (Kalman filter) for the system linearized about the origin and plot the state estimate and covariance of each state (using error bars) for a trajectory starting at the origin and for a duration of $K = 20$.

(b) Construct a moving horizon estimator for the system using time windows of length $N = 1$, $N = 3$, and $N = 6$ and compare the state estimate to the result from (a).

(c) Change the penalty on the initial state in the window from $P[0]$ to the value of P obtained from the steady state estimator for the linearized system and compare the performance for the three horizon lengths in (b).

(d) Suppose that the disturbance V is constrained to take values in the the range -0.1 to 0.1. Compare the steady state optimal estimator for the linearized system to a moving horizon estimator with appropriate horizon that takes the constraints into account. How does the moving horizon estimator compare to the linearized estimator?

Problem 3. Consider the PVTOL system (`pvtol_noisy`) used throughout the course:

$$\begin{aligned} m\ddot{x} &= F_1 \cos\theta - F_2 \sin\theta - c\dot{x} + D_x, \\ m\ddot{y} &= F_1 \sin\theta + F_2 \cos\theta - c\dot{y} - mg + D_y, \\ J\ddot{\theta} &= rF_1, \end{aligned} \qquad \vec{Y} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix}.$$

Assume that the input disturbances are modeled by independent, first order Markov (Ornstein-Uhlenbeck) processes with $Q_D = \operatorname{diag}(0.01, 0.01)$ and $\omega_0 = 1$ and that the noise is modeled as white noise with covariance matrix

$$Q_N = \begin{bmatrix} 2 \times 10^{-4} & 0 & 1 \times 10^{-5} \\ 0 & 2 \times 10^{-4} & 1 \times 10^{-5} \\ 1 \times 10^{-5} & 1 \times 10^{-5} & 1 \times 10^{-4} \end{bmatrix}.$$

Design a controller consisting of a trajectory generation module, a gain-scheduled, trajectory tracking module, and a state estimation module the moves the system from the origin to the equilibrum point point x_f, $y_\text{f} = 10, 0$ while satisfying the constraint $0.5\sin(\pi x/10) - 0.1 \leq y \leq 1$.

Provide enough detail of the controller architecturee and design approach to demonstrate your understanding of the individual elements of the control design as well as their interconnection.

(A template that can be used to generate the plots required for this problem is availalble on the course website.)