

CALIFORNIA INSTITUTE OF TECHNOLOGY  
Computing and Mathematical Sciences

CDS 112/Ae 103b

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Homework Set #8

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Due: 1 Mar 2023

**Problem 1.** Suppose that we wish to estimate the position of a particle that is undergoing a random walk in one dimension (i.e., along a line). We model the position of the particle as

$$x[k+1] = x[k] + u[k],$$

where  $x$  is the position of the particle and  $u$  is a white noise processes with  $E\{u[i]\} = 0$  and  $E\{u[i]u[j]\} = R_u\delta(i-j)$ . We assume that we can measure  $x$  subject to additive, zero-mean, Gaussian white noise with covariance 1.

(a) Construct a Kalman filter to estimate the position of the particle given the noisy measurements of its position. Compute the steady-state expected value and covariance of the error of your estimate.

(b) Suppose that  $E\{u[0]\} = \mu \neq 0$  but is otherwise unchanged. How would your answer to part (a) change?

**Problem 2.** Consider a simple spring mass system that has two sensors: a position sensor (based on GPS) and a velocity sensor. In this problem we will use the predictor-corrector form of the discrete-time Kalman filter equations to simulate the convergence of the covariance estimate of the position and velocity of the mass.

The dynamics of the system are given by

$$m\ddot{q} + b\dot{q} + kq = v$$

where  $m = 10$  g is the mass of the system,  $b = 0.1$  N/m/s is the damping coefficient, and  $k = 1$  N/m is the spring constant. We assume that  $v$  consists of white noise with intensity 0.1 N.

Assume that the position measurement is done at 5 Hz with an accuracy (standard deviation of each estimate) of 1 m and that the velocity measurement is done at 20 Hz with an accuracy of 0.1 m/s.

Convert the system to the equivalent discrete-time system, and compute the time history of the error covariance for the optimal estimate in predictor-corrector form. Plot the diagonal elements of  $P_q[k|k]$  and  $P_{\dot{q}}[k|k]$ , corresponding to the accuracy of the position and velocity estimate.

(To handle the fact that the initial estimate is not known, assume that the estimator is initialized with a (noisy) measurement from the position and velocity sensors.)

**Problem 3.** Consider the problem of estimating the position of a car operating on a road whose dynamics are modeled as described in Example 2.3 in OBC. We assume that the car is executing a lane change maneuver and we wish to estimate its position using a set of available sensors:

- A stereo camera pair, which has relatively poor longitudinal ( $x$ ) accuracy but good lateral position ( $y$ ) accuracy. We model the covariance of the sensor noise as  $R_{\text{lat}} = \text{diag}(1, 0.1)$ .
- An automotive grade radar, which has good longitudinal position ( $x$ ) accuracy but poor lateral ( $y$ ) accuracy, with  $R_{\text{lon}} = \text{diag}(0.1, 1)$ .
- We assume the radar can also measure the longitudinal velocity ( $\dot{x}$ ) as an optional measurement, with  $R_{\text{vel}} = 1$ .

In this problem we assume that the detailed model of the system is not known and also that the inputs to the vehicle (throttle and steering) are not known. We use a variety of system models to explore how these different measurements can be fused to obtain estimates and predictions of the vehicle position.

(a) Consider a model of the vehicle consisting of a particle in 2D, with the velocity of particle in the  $x$  and  $y$  direction taken as the input:

$$\dot{x} = u_1, \quad \dot{y} = u_2$$

A discrete-time version of the system dynamics is given by

$$x[k+1] = x[k] + u_1[k] * T_s, \quad y[k+1] = y[k] + u_2[k] * T_s,$$

where  $T_s = 0.1$  s is the sampling time between sensor measurements.

Construct an estimator for the system using a combination of the stereo pair and the radar (position only). Estimate the state and covariance of the system during the lane change maneuver from Example 2.3 and predict the state for the next 4 seconds.

(b) Assume now that we now add (noisy) measurement of the velocity from the radar as an approximation of the input  $u_1$ . Update your Kalman filter to utilize this measurement (with no filtering), and replot the estimate and prediction for the system.

(c) To provide a better prediction, we can increase the complexity of our model so that it includes the velocity of the vehicle as a state, allowing us to model the acceleration as the input. In continuous time, this model is given by

$$\ddot{x} = u_1, \quad \dot{y} = u_2$$

(note that we are still modeling the lateral position using a single integrator).

Convert this model to discrete time and construct an estimator for the system using a combination of the stereo pair and the radar (position and velocity). Estimate the state and covariance of the system during the lane change maneuver and predict the state for the next 4 seconds.

Note: in this problem you have quite a bit of freedom in how you model the disturbances, which should model the unknown inputs to the vehicle being observed. Make sure to provide some level of justification for how you chose these disturbances.