CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 112/Ae 103b

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Homework Set #5

Issued: 1 Feb 2023
Due: 8 Feb 2023

Problem 1. Consider a nonlinear control system

$$\dot{x} = f(x, u)$$

with linearization

$$\dot{x} = Ax + Bu.$$

Show that if the linearized system is reachable, then there exists a (local) control Lyapunov function for the nonlinear system. (Hint: start by proving the result for a stable system.)

Problem 2. [Instability of MPC with short horizons (Mark Cannon, Oxford University, 2020)] Consider a linear, discrete time system with dynamics

$$x[k+1] = \begin{bmatrix} 1 & 0.1 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} u[k], \qquad y[k] = \begin{bmatrix} 1 & 0 \end{bmatrix} x[k]$$

with finite time horizon cost given by

$$J(x[k], u) = \sum_{i=0}^{N-1} \Big(y^2[k+i] + u^2[k+i] \Big) + y^2[k+N].$$

(a) Show that the predicted state of the system can be written in the form

$$\begin{bmatrix} x[k] \\ x[k+1] \\ \vdots \\ x[k+N] \end{bmatrix} = \mathcal{M} x[k] + \mathcal{L} \begin{bmatrix} u[k] \\ u[k+1] \\ \vdots \\ u[k+N-1] \end{bmatrix}$$

and give formulas for $\mathcal{M}$ and $\mathcal{L}$ in terms of A, B, and C for the case $N = 3$.

(b) Show that the cost function can be written as

$$J(x[k], u) = \bar{u}^\mathsf{T}[k] H \bar{u}[k] + 2x^\mathsf{T}[k] F \bar{u}[k] + x^\mathsf{T}[k] G x[k],$$

where $\bar{u}[k] = (u[k], u[k+1], \dots, u[k+N-1]]$, and give expressions for F, G, and H.

(c) Show that the RHC controller that minimizes the cost function for a horizon length of N can be written as $u = -Kx$ and find an expression for K in terms of F, G, and H. Show that for $N = 3$ the feedback gain is given by

$$K = \begin{bmatrix} 0.1948 & 0.1168 \end{bmatrix}.$$

(d) Compute the closed loop eigenvalues for the system with a receding horizon controller with $N = 3$ and show that the system is unstable. What is the smallest value of N such that the system is stable?

(e) Change the terminal cost to use the optimal cost-to-go function returned by the `dlqr` command in MATLAB or Python. Verify that the closed loop system is stable for $N = 1, \dots, 5$.

Problem 3. Consider the dynamics of the vectored thrust aircraft described in Examples 2.4 and 3.5. The equations of motion are given by

$$\begin{aligned} m\ddot{x} &= F_1 \cos\theta - F_2 \sin\theta - c\dot{x}, \\ m\ddot{y} &= F_1 \sin\theta + F_2 \cos\theta - c\dot{y} - mg, \\ J\ddot{\theta} &= rF_1. \end{aligned} \tag{1}$$

with parameter values

$$m = 4 \text{ kg}, \qquad J = 0.0475 \text{ kg m}^2, \qquad r = 0.25 \text{ m}, \qquad g = 9.8 \text{ m/s}^2, \qquad c = 0.05 \text{ Ns/m},$$

which corresponds roughly to the values for the Caltech ducted fan flight control testbed. Assume that the inputs must satisfy the constraints

$$|F_1| \leq 0.1\,|F_2|, \qquad 0 \leq F_2 \leq 1.5\,mg.$$

(a) Design a receding horizon controller for the system that stabilizes the origin using an optimization horizon of $T = 3$ s and an update period of $\Delta T = 1$ s. Demonstrate the performance of your controller from initial conditions starting at initial position $(x_0,\, y_0) = (0 \text{ m}, 5 \text{ m})$ and desired final position $(x_\text{f},\, y_\text{f}) = (10 \text{ m}, 5 \text{ m})$ (all other states should be zero).

(b) Suppose that the system is subject to a sinusoidal disturbance force due to wind blowing in the horizontal direction, so that the dynamics in the x coordinate become

$$m\ddot{x} = F_1 \cos\theta - F_2 \sin\theta - c\dot{x} + d$$

with $d = \sin(t)$. Design a two-layer (inner/outer) feedback controller that uses the trajectories from (a) as inputs to an LQR controller that provides disturbance rejection. Compare the performance of the RHC controller in (a) in the presence of the disturbance to a two-layer controller using the initial and final conditions from (a).

(The Python function `pvtol-windy` in `pvtol.py` provides a model of this system using a third input corresponding to d.)