CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 112/Ae 103b

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Problem 1. Consider the normalized, linearized inverted pendulum which is described by

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u = Ax + Bu.$$

Find a state feedback that minimizes the quadratic cost function

$$J = \int_0^\infty \left(q_1 x_1^2 + q_2 x_2^2 + q_u u^2\right) dt$$

where $q_1 \geq 0$ is the penalty on position, $q_2 \geq 0$ is the penalty on velocity, and $q_u > 0$ is the penalty on control actions. Show that the closed loop characteristic polynomial has the form $s^2 + 2\zeta_0\omega_0\, s + \omega_0^2$ and that $\omega_0 \geq 1$ and $\zeta_0 \geq \sqrt{2}/2$.

Problem 2. Consider the dynamics of the vectored thrust aircraft described in Examples 2.4 and 3.5. The equations of motion are given by

$$\begin{aligned} m\ddot{x} &= F_1 \cos\theta - F_2 \sin\theta - c\dot{x}, \\ m\ddot{y} &= F_1 \sin\theta + F_2 \cos\theta - c\dot{y} - mg, \\ J\ddot{\theta} &= rF_1. \end{aligned} \tag{1}$$

with parameter values

$$m = 4 \text{ kg}, \quad J = 0.0475 \text{ kg m}^2, \quad r = 0.25 \text{ m}, \quad g = 9.8 \text{ m/s}^2, \quad c = 0.05 \text{ Ns/m},$$

which corresponds roughly to the values for the Caltech ducted fan flight control testbed.

(a) Design a feasible trajectory (x_d, u_d) for the system that corresponds to moving to the right by 10 meters over a period of 5 seconds. (For the purpose of designing this trajectory, you can assume $c = 0$ so that the system is differentially flat.) Plot the open loop response of the system (with $c \neq 0$) when the desired input u_d is applied.

(b) Design a time-invariant linear controller for the system that attempts to track the trajectory using pure feedback, of the form $u = u_\text{eq} - K(x - x_\text{d})$, where u_eq is the input required to hold the system at hover (replacing u_d). Plot the response of the system along with the inputs.

(c) Add back in the feedforward term u_d and compare the performance (errors) of the pure feedback controller ($u = u_\text{eq} - K(x - x_\text{d})$) with the controller including feedforward ($u = u_\text{d} - K(x - x_\text{d})$).

(d) Suppose that the system is subject to a disturbance force due to wind blowing from the right, so that the dynamics in the x coordinate become

$$m\ddot{x} = F_1 \cos\theta - F_2 \sin\theta - c\dot{x} + d$$

with $d = -5$. Design a linear feedback controller using integral action on the x and y errors and compare the performance of the controller with integral action to the controller with feedforward and feedback but no integral action.

(The Python function `pvtol-windy` in `pvtol.py` provides a model of this system using a third input corresponding to d. Note that you cannot use `create_statefbk_iosystem` in this case, so you'll have to construct the controller with integral action manually.)

Problem 3. Consider a nonlinear control system with gain scheduled feedback

$$\dot{e} = f(e, v) \qquad v = k(\mu)e,$$

where $\mu(t) \in \mathbb{R}$ is an externally specified parameter (e.g., the desired trajectory) and $k(\mu)$ is chosen such that the linearization of the closed loop system around the origin is stable for each fixed μ.

Show that if $|\dot{\mu}|$ is sufficiently small then the equilibrium point is locally asymptotically stable for the full nonlinear, time-varying system. (Hint: find a Lyapunov function of the form $V = x^\mathsf{T} P(\mu) x$ based on the linearization of the system dynamics for fixed μ and then show this is a Lyapunov function for the full system.)