Problem 1. Consider the normalized, linearized inverted pendulum which is described by
\[
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = Ax + Bu.
\]
Find a state feedback that minimizes the quadratic cost function
\[
J = \int_0^\infty (q_1 x_1^2 + q_2 x_2^2 + q_u u^2) \, dt
\]
where \(q_1 \geq 0\) is the penalty on position, \(q_2 \geq 0\) is the penalty on velocity, and \(q_u > 0\) is the penalty on control actions. Show that the closed loop characteristic polynomial has the form \(s^2 + 2\zeta_0 \omega_0 s + \omega_0^2\) and that \(\omega_0 \geq 1\) and \(\zeta_0 \geq \sqrt{2}/2\).

Problem 2. Consider the dynamics of the vectored thrust aircraft described in Examples 2.4 and 3.5. The equations of motion are given by
\[
\begin{align*}
mx &= F_1 \cos \theta - F_2 \sin \theta - cx, \\
y &= F_1 \sin \theta + F_2 \cos \theta - cy - mg, \\
J \ddot{\theta} &= rF_1,
\end{align*}
\]
with parameter values
\[
m = 4 \text{ kg}, \quad J = 0.0475 \text{ kg m}^2, \quad r = 0.25 \text{ m}, \quad g = 9.8 \text{ m/s}^2, \quad c = 0.05 \text{ Ns/m},
\]
which corresponds roughly to the values for the Caltech ducted fan flight control testbed.

(a) Design a feasible trajectory \((x_d, u_d)\) for the system that corresponds to moving to the right by 10 meters over a period of 5 seconds. (For the purpose of designing this trajectory, you can assume \(c = 0\) so that the system is differentially flat.) Plot the open loop response of the system (with \(c \neq 0\) when the desired input \(u_d\) is applied.

(b) Design a time-invariant linear controller for the system that attempts to track the trajectory using pure feedback, of the form \(u = u_{eq} - K(x - x_d)\), where \(u_{eq}\) is the input required to hold the system at hover (replacing \(u_d\)). Plot the response of the system along with the inputs.

(c) Add back in the feedforward term \(u_d\) and compare the performance (errors) of the pure feedback controller \((u = u_{eq} - K(x - x_d))\) with the controller including feedforward \((u = u_d - K(x - x_d))\).

(d) Suppose that the system is subject to a disturbance force due to wind blowing from the right, so that the dynamics in the \(x\) coordinate become
\[
m\ddot{x} = F_1 \cos \theta - F_2 \sin \theta - cx + d
\]
with \( d = -5 \). Design a linear feedback controller using integral action on the \( x \) and \( y \) errors and compare the performance of the controller with integral action to the controller with feedforward and feedback but no integral action.

(The Python function `pvtol-windy` in `pvtol.py` provides a model of this system using a third input corresponding to \( d \). Note that you cannot use `create_statefbk_iosystem` in this case, so you’ll have to construct the controller with integral action manually.)

**Problem 3.** Consider a nonlinear control system with gain scheduled feedback

\[
\dot{e} = f(e, v) \quad v = k(\mu)e,
\]

where \( \mu(t) \in \mathbb{R} \) is an externally specified parameter (e.g., the desired trajectory) and \( k(\mu) \) is chosen such that the linearization of the closed loop system around the origin is stable for each fixed \( \mu \).

Show that if \( |\dot{\mu}| \) is sufficiently small then the equilibrium point is locally asymptotically stable for the full nonlinear, time-varying system. (Hint: find a Lyapunov function of the form \( V = x^TP(\mu)x \) based on the linearization of the system dynamics for fixed \( \mu \) and then show this is a Lyapunov function for the full system.)