CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

# CDS 112/Ae 103b

R. Murray | **Homework Set #3** | Issued: 18 Jan 2023
Winter 2023 | | Due: 25 Jan 2023

**Problem 1** (optional). This problem is completely optional and will not be used in determining your grade for this problem set. I recommend working through it if you finish the other problems on the set and have spent less than 9 hours total on the course this week.

(a) Let $G_1, G_2, \ldots, G_k$ be a set of row vectors on $\mathbb{R}^n$. Let $F$ be another row vector on $\mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$ satisfying $G_i x = 0$, $i = 1, \ldots, k$, we have $Fx = 0$. Show that there are constants $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that

$$F = \sum_{i=1}^{k} \lambda_i G_i.$$

(b) Let $x^* \in \mathbb{R}^n$ be an the extremal point (maximum or minimum) of a function $f$ subject to the constraints $g_i(x) = 0$, $i = 1, \ldots, k$. Define a new function

$$\tilde{f}(x, \lambda) = f(x) + \sum_{i=1}^{k} \lambda_i g_i(x).$$

Assuming that the gradients $\partial g_i(x^*)/\partial x$ are linearly independent, show that there are $k$ scalars $\lambda_i$, $i = 1, \ldots, k$ such that $x^*$ is the (unconstrained) extremal of $\tilde{f}(x, \lambda)$. (Hint: you will need part (a) at some point.)

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**Problem 2.** Consider the optimal control problem for the system

$$\dot{x} = -ax + bu,$$

where $x = \mathbb{R}$ is a scalar state, $u \in \mathbb{R}$ is the input, the initial state $x(t_0)$ is given, and $a, b \in \mathbb{R}$ are positive constants. (Note that this system is not quite the same as the one in Example 3.2.) The cost function is given by

$$J = \tfrac{1}{2} \int_{t_0}^{t_\text{f}} u^2(t)\, dt + \tfrac{1}{2} c x^2(t_\text{f}),$$

where the terminal time $t_\text{f}$ is given and $c$ is a constant.

(a) Solve explicitly for the optimal control $u^*(t)$ and the corresponding state $x^*(t)$ in terms of $t_0$, $t_\text{f}$, $x(t_0)$ and $t$ and describe what happens to the terminal state $x^*(t_\text{f})$ as $c \to \infty$.

Hint: Once you have $u^*(t)$, you can use the convolution equation to solve for the optimal state $x^*(t)$.

(b) Let $a = 1$, $b = 1$, $c = 1$, and $t_\text{f} - t_0 = 1$. Solve the optimal control problem numerically using python-control (or MATLAB) and compare it to your analytical solution by plotting the state and input trajectories for each solution. Take the initial conditions as $x(t_0) = 1$, 5, and 10.

(c) Suppose that we wish to have the final state be exactly zero. Change the optimization problem to impose a final constraint instead of a final cost. Solve the fixed endpoint, optimal control problem using python-control (or MATLAB), and plot the state and input trajectories for the initial conditions in 2b.

**Problem 3.** Consider the dynamics of the vectored thrust aircraft described in Examples 2.4 and 3.5. The equations of motion are given by

$$\begin{aligned} m\ddot{x} &= F_1 \cos\theta - F_2 \sin\theta - c\dot{x}, \\ m\ddot{y} &= F_1 \sin\theta + F_2 \cos\theta - c\dot{y} - mg, \\ J\ddot{\theta} &= rF_1. \end{aligned} \tag{1}$$

with parameter values

$$m = 4 \text{ kg}, \quad J = 0.0475 \text{ kg m}^2, \quad r = 0.25 \text{ m}, \quad g = 9.8 \text{ m/s}^2, \quad c = 0.05 \text{ Ns/m},$$

which corresponds roughly to the values for the Caltech ducted fan flight control testbed.

We wish to generate an optimal trajectory for the system that corresponds to moving the system for an initial hovering position to a hovering position one meter to the right ($x_\text{f} = x_0 + 1$).

For each of the parts below, you should solve for the optimal input, simulate the (open loop) system, and plot the $xy$ trajectory of the system, along with the angle $\theta$ and inputs $F_1$ and $F_2$ over the time interval. In addition, create a time and record the following information for each approach:

- the computation time required;
- the final position (for the open loop system);
- the weighted integrated cost of the input along the trajectory:

$$\int_0^T \left(10F_1^2(\tau) + (F_2 - mg)^2(\tau)\right) d\tau. \tag{2}$$

(a) Solve for an optimal trajectory using a quadratic cost from the final point with weights

$$Q_x = \text{diag}([1, 1, 10, 0, 0, 0]), \qquad Q_u = \text{diag}([10, 1]).$$

This cost function attempts to minimize the angular deviation $\theta$ and the sideways force $F_1$.

(b) Re-solve the problem using Bezier curves as the basis functions for the inputs. This should give you smoother inputs and a nicer response.

(c) Re-solve the problem using a terminal cost $V(x(T)) = x(T)^\mathsf{T} P_1 x(T)$ to try to get the system closer to the final value. You should try adjusting the cost along the trajectory $Q_x$ versus the terminal cost $P_1$ to minimize the weighted integrated cost (2).

(d) Re-solve the problem using a terminal *constraint* to try to get the system closer to the final value. Adjust the cost along the trajectory to try to minimize the cost in equation (2). (Hint: you may have to use an initial guess to get the optimization to converge.)

(e) If $c = 0$, it can be shown that this system is differentially flat (see Example 2.4). Setting $c = 0$, re-solve the optimization problem using differential flatness. (The flatness mappings can be found in the file `pvtol.py`, available on the course website.)