CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 112/Ae 103b

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Problem 1. Consider the trajectory generation problem for the system

$$\frac{dx}{dt} = -ax^3 + bu,$$

where $x \in \mathbb{R}$ is a scalar state, $u \in \mathbb{R}$ is the input, the initial state $x(t_0)$ is given, and $a, b \in \mathbb{R}$ are positive constants.

(a) Show that the system is differentially flat with appropriate choice of output(s) and compute the state and input as a function of the flat output(s).

(b) Using the polynomial basis $\{t^k,\, k = 0, \dots, M\}$ with an appropriate choice of M, solve for the (non-optimal) trajectory between $x(t_0)$ and $x(t_\text{f})$. Your answer should specify the explicit input $u_\text{d}(t)$ and state $x_\text{d}(t)$ in terms of t_0, t_f, $x(t_0)$, $x(t_\text{f})$, and t.

(c) Suppose that we also wish to constrain the inputs and the initial and final time, so that $u(t_0)$ and $u(t_\text{f})$ are specified. Show how to extend your answer from part (b) to this case. (If your answer in part (b) already involved specification of the input at the initial and final times, then try specifying $\dot{u}$ at the initial and final times).

Problem 2. Consider the nonholonomic integrator in Example 2.2 in OBC:

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = u_2, \qquad \dot{x}_3 = x_2 u_1.$$

Using Bezier curves as the basis functions for the flat outputs, find a trajectory between $x_0 = (0, 0, 0)$, $u_0 = (1, 0)$ and $x_\text{f} = (10, 0, 5)$, $u_\text{f} = (1, 1)$ over a time interval $T = 2$. Plot the states and inputs for your trajectory.

Problem 3. Consider a predator-prey system with forcing whose dynamics are given by a modified version of the Lotka-Volterra equations:

$$\frac{dx_1}{dt} = (\alpha + u)x_1 - \beta x_1 x_2, \qquad \frac{dx_2}{dt} = \delta x_1 x_2 - \gamma x_2.$$

In this equation, x_1 represents the prey population, x_2 represents the predator population, and u is an input. The parameters α, β, δ, and γ are all positive and we take our time unit to be in years.

(a) Find the possible equilibrium points (x_e, u_e) that correspond to positive predator and prey populations ($x_{\text{e},1} > 0$, $x_{\text{e},2} > 0$), and determine whether or not we can asymptotically stabilize the resulting equilibrium point(s) using state feedback.

(b) Show that the system is differentially flat and computing the mappings between the states and inputs and the flat flag.

(c) Using the parameter values $\alpha = 1.1$, $\beta = 0.4$, $\delta = 0.1$, $\gamma = 0.4$, find a feasible trajectory for the system from the equilibrium point $x_0 = (4, 2.75)$ to the equilibrium point $x_\text{f} = (4, 25)$ over a period of $T = 2$ years. Plot the trajectories of the populations of each species and the input over the time horizon, as well as the prey versus predator population (with the prey on the x axis and predator on the y axis).

(d) Suppose that we introduce uncertainty into the model by modifying the predator death rate γ to take the value $\gamma' = 1.5\gamma$. Design a state feedback controller that stabilizes the trajectory computed in part 3c and plot the open and closed loop trajectories as functions of time as well as the prey versus predator population.

Problem 4. A simplified model of the steering control problem is described in FBS2e, Example 6.13. The lateral dynamics can be approximated by the (normalized) linearized dynamics

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} \gamma \\ 1 \end{bmatrix} u, \qquad y \; = x_1,$$

where $x = (y, \theta) \in \mathbb{R}^2$ is the state of the system, γ is a parameter that depends on the forward speed of the vehicle, and y is the lateral position of the vehicle.

Suppose that we wish to track a piecewise constant reference trajectory that consists of moving left and right by 1 meter:

$$x_\text{d} = \begin{bmatrix} \texttt{square}(2\pi t/20) \\ 0 \end{bmatrix}, \qquad u_\text{d} = 0,$$

where `square` is the square wave function in `scipy.signal`. Suppose further that the speed of the vehicle varies such that the parameter γ satisfies the formula

$$\gamma(t) = 2 + 2\sin(2\pi t/50).$$

(a) Show that the desired trajectory given by x_d and u_d satisfy the dynamics of the system at all points in time except the switching points of the square wave function.

(b) Suppose that we fix $\gamma = 2$. Use eigenvalue placement to design a state space controller $u = u_\text{d} - K(x - x_\text{d})$ where the gain matrix K is chosen such that the eigenvalues of the closed loop poles are at the roots of $s^2 + 2\zeta\omega_0 s + \omega_0^2$, where $\zeta = 0.7$ and $\omega_0 = 1$. Apply the controller to the time-varying system where $\gamma(t)$ is allowed to vary and plot the output of the system compared to the desired output.

(c) Find gain matrices K_1, K_2, and K_3 corresponding to $\gamma = 0, 2, 4$ and design a gain-scheduled controller that uses linear interpolation to compute the gain for values of γ between these values. Compare the performance of the gain scheduled controller to your controller from part (b).