CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 112/Ae 103b

R. Murray | **Homework Set #1** | Issued: 4 Jan 2023
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Problem 1. Consider a second order linear system with dynamics given by the following state space dynamics and transfer function:

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\-k & -b\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u, \qquad y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}, \qquad P(s) = \frac{1}{s^2+bs+k}.$$

In this problem you will design a controller for this system in either the time or frequency domain (depending on which you are most comfortable with).

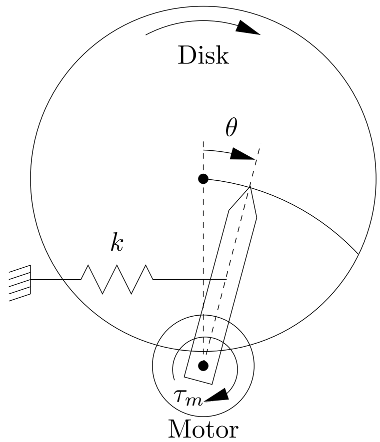
(a) Design either a state space or frequency domain controller for the system that can be used to track a reference signal r (corresponding to a desired state $x_\text{d} = (r, 0)$). Write down the closed loop dynamics for the system and give conditions on the parameters of your controller such that the closed loop system is stable and the steady state error ($e = y - r$) for a step input of magnitude 1 is no more than γ. (The conditions for the gains in your controller should be in terms of inequalities involving the system parameters b and k and performance parameter γ.)

(b) Suppose now that we take $k = 1$ and $b = 0.1$. Pick specific parameters for your controller such that the steady state error is no more than 10% ($\gamma = 0.1$) and the settling time is more more than 5 seconds. Plot the step response for the sytem and compute the rise time, settling time, overshoot, and steady state error for your design in response to a step change in the input r. (You can do the computations either analytically or computationally.)

(c) Using the same parameters for the system and your controller, compute the steady state ratio of the output magnitude to the reference magnitude and the phase offset between the output and the reference for a reference signal $r = \sin(2t)$. (You can do these computations either analyitically or computationally.)

If you carry out the computations for parts 1b and/or 1c numerically, include the MATLAB or Python code used to generate your results, as well as any plots generated by your code and used to determine your answers.

Problem 2. Consider a simple mechanism for positioning a mechanical arm and the associated equations of motion:



$$J\ddot{\theta} = -b\dot{\theta} - kr\sin\theta + \tau_{\mathrm{m}}$$

The system consists of a spring-loaded arm that is driven by a motor. The motor applies a force against the spring and pulls the tip across a rotating platter. The input to the system is the motor torque τ_{m}. In the diagram above, the force exerted by the spring is a nonlinear function of the head position due to the way it is attached. The output of the system sensors is the offset of the end of the arm from the center of the platter, with a small offset depending on the angular rate:

$$y = l\theta + \epsilon\dot{\theta}$$

Take the system parameters to be

$$k = 1, \quad J = 100, \quad b = 10, \quad r = 1, \quad l = 2, \quad \epsilon = 0.01.$$

Starting with the template Jupyter notebook posted on the course website, create a Jupyter notebook that documents the following operations:

(a) Compute the linearization of the dynamics about the equilibrium point corresponding to $\theta_{\mathrm{e}} = 15^\circ$.

(b) Plot the step response of the linearized, open-loop system and compute the rise time and settling time for the output y.

(c) Assuming that the full system state is available, design a state feedback controller for the system that allows the system to track a desired position y_{d} and sets the closed loop eigenvalues to $\lambda_{1,2} = -10 \pm 10i$. Plot the step response for the closed loop system and compute the rise time, settling time, and steady state error for the output y.

(d) Plot the frequency response (Bode plot) of the closed loop system. Use the frequency response to compute the steady state error for a step input and the bandwidth of the system (frequency at which the magnitude of the output is less than $1/\sqrt{2}$ from its reference value).

Hint: if you are not familiar with frequecy responses of linear time invariant systems, see Section 6.3 (Input/Output Response) of FBS2e.

(e) Create simulations of the full nonlinear system with the linear controllers designed in part 2c and plot the response of the system from an initial position of 0 m at $t = 0$, to 1 m at $t = 0.5$, to 3 m at $t = 1$, to 2 m at $t = 1.5$.