CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 112/Ae 103a

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Winter 2022 | | Due: 9 Mar 2022

Problem 1. Conserving fuel is critical in satellite attitude control. The simplified single-axis satellite dynamics can be described by

$$I\ddot{\theta} = u$$

where I is the rotational inertia and the input u has maximum magnitude $N = 1$. To keep the analysis simple, use $I = 1$. The desired maneuver is described by the constraints $x(0) = x_0$ and $x(T) = 0$ where the state vector is $x = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^T$. The minimum fuel cost function is

$$J(u) = \int_0^T (1 + \alpha|u|)dt$$

where the parameter $\alpha \geq 0$ is a weighting factor that trades off maneuver time for fuel ($\alpha = 0$ gives minimum time, while $\alpha \to \infty$ gives minimum fuel, and hence the maneuver takes infinite time.)

(a) [5 pts] Write down a state-space representation for the system and the Hamiltonian for this problem. Write the necessary conditions for optimality using Pontryagin's maximum principle, including boundary conditions.

(b) [10 pts] Show that the optimal control for this problem is a bang-deadzone-bang type of input, and write down the switching conditions. What is the maximum number of switches from one value of control to another during the maneuver?

(c) [5 pts] Sketch the time-history of $u(t)$, $\dot{\theta}(t)$ and $\theta(t)$ for a maneuver where $\dot{\theta}(0) = 0$ and $\theta(0) > 0$. (You do not need a computer; it is sufficient to sketch the general shape of the signals.)

Problem 2. In this problem you will derive the optimal LQR feedback for a discrete-time system

$$x_{k+1} = Ax_k + Bu_k$$

with cost

$$J(n, N) = \sum_{k=n}^{N} \left(x_k^T Q x_k + u_k^T R u_k \right) \triangleq x_n^T P_n x_n$$

(a) [5 pts] Set $u_k = 0$, and for the unforced system $x_{k+1} = Ax_k$ derive the difference equation for P_n as a function of P_{n+1}, Q and A (this is the discrete-time version of the Lyapunov equation). What is the boundary condition on the difference equation for P_n?

(Hint: write two different expressions for $J(n + 1, N) - J(n, N)$.)

(b) [5 pts] Assuming linear state feedback $u_k = K_k x_k$, what is the cost $J(n, N)$ for *any* (not necessarily optimal) state feedback K_k?

(c) [10 pts] The optimal linear control at time-step n can be obtained by minimizing the trace of P_n for a given P_{n+1}. Use the fact that

$$\frac{\partial(\text{trace}(K^T X))}{\partial K} = \frac{\partial(\text{trace}(X^T K))}{\partial K} = X$$

to write down the discrete-time Riccati equation (i.e. an equation for P_n that does not depend explicitly on K) and the optimal gain K_n.

(d) [5 pts] A continuous time system $\dot{x} = Ax + Bu$ can be approximated for sufficiently small sample interval T by

$$\frac{x_{k+1} - x_k}{T} = Ax_k + Bu_k$$

The cost for a continuous time system can be approximated as

$$J = \int \left(x^T Q x + u^T R u\right) \simeq T \sum \left(x^T Q x + u^T R u\right)$$

Derive the Riccati equation for the continuous time system by substituting the discrete-time approximations for the continuous system into the equation you derived in part (c) and taking the limit for $T \to 0$.

Problem 3. This problem shows that given any state space control law, we can obtain that same control law using a receding horizon formulation with appropriately chosen cost functions. Consider a linear control system

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n, u \in \mathbb{R}$$

with a state space controller $u = -Kx$ that stabilizes the origin.

(a) (5 points) Show that there exists a positive semi-definite matrix Q_x and a positive definite matrix Q_u such that $u = -Kx$ is optimal with respect to the cost function

$$J(x) = \int_0^\infty x^T Q_x x + u^T Q_u u \, d\tau.$$

(Hint: Use the fact that for the optimal feedback, $B^T P = Q_u K$ and use this to eliminate the quadratic term in the Riccati equation. You should be left with an equation that you can solve with a bit of cleverness. If you can't get this part, go on to part (b) assuming that the statement is true.)

(b) (5 points) Assuming part (a), show that there exists a positive semi-definite matrix Q_x, a positive definite matrix Q_u and a positive definite matrix P_T such that $u = -Kx$ is the optimal control with respect to the receding horizon control

$$u_{[t,t+T]} = \arg\min \int_t^{t+T} x^T Q_x x + u^t Q_u u \, d\tau + x^T(T) P_T x(T),$$

where we apply $u_{[t,t+T]}(t)$ at each time t. (Hint: Look back at the lecture notes on quadratic regulators from week 6 to find the form of the optimal control in the finite time case. Then make use of the fact that you get to choose P_T.)

Problem 4. Consider the nonlinear control system

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{x}_3 &= x_2 u_1 - x_1 u_2, \end{aligned} \qquad \begin{aligned} &x_1, x_2, x_3 \in \mathbb{R}, \\ &u_1, u_2 \in \mathbb{R}. \end{aligned}$$

In this problem we will explore different methods for generating feasible trajectories for the system using differential flatness and optimal control.

(a) [5 pts] Show that the system is differentially flat using the following flat outputs:

$$z_1 = x_1, \qquad z_2 = x_3 + x_1 x_2.$$

Compute the states and inputs as a function of the flat outputs and their derivatives.

(b) [5 pts] Compute a feasible trajectory from $x_0 = (0, 0, 0)$ to $x_f = (0, 0, 1)$ in time $T = 1$. (Note: you can use MATLAB to carry out numerical computations, but you can also solve this one easily by hand if you organize your calculation a bit.)

(c) [5 pts] Compute the necessary conditions for a feasible trajectory between an initial point x_0 and a final point x_f to minimize the cost function

$$J = \frac{1}{2} \int_0^T u_1^2 + u_2^2 \, dt,$$

where $T = 1$.

(d) [5 pts] Show that controls of the form

$$u_1(t) = r \sin(\omega t + \phi), \qquad u_2(t) = r \cos(\omega t + \phi)$$

satisfy the necessary conditions, where r, ω and ϕ are all constants.

(e) [5 pts] Compute an explicit optimal trajectory from $x_0 = (0, 0, 0)$ to $x_1 = (0, 0, 1)$ in time $T = 1$. What is the cost? Is it unique?

Problem 5. For each of the statements below, indicate whether the statement is true or false. If you indicate the statement is true, give a justification for your answer (e.g, a short derivation or proof). If you indicate the statement is false, include a counter-example. [1 point each for getting true/false correct + 1 point for derivation/counter-example).

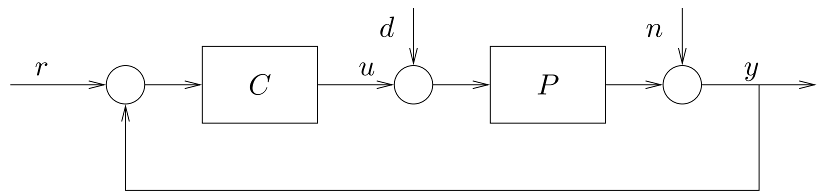
(a) If a linear system is reachable, there exists a unique flat output.

(b) If a control system has constraints on the inputs, all optimal trajectories correspond to "bang-bang" controllers.

(c) The solution of the infinite-horizon, linear quadratic regulator problem always gives a stable closed loop system for $Q_x, Q_u > 0$.

(d) All reachable linear systems can be stabilized using receding horizon control with a time horizon $T = 1$ and an update period of $\delta = 0.1$.

Problem 6. Consider the following control system



where

$$P(s) = \frac{\gamma}{s}.$$

A state space representation for P is given by

$$\dot{x} = ku + d$$
$$y = x + n.$$

We wish to design the control law $C(s)$ using an observer-based optimal control law.

(a) (5 pts) Design a state feedback controller, $u = -Kx$, for the system that minimizes the cost function

$$J = \int_0^\infty y^2 + u^2 \, dt$$

(assume that the full state is available for feedback and ignore the disturbances and noise).

(b) (5 pts) Design an observer for this system that minimizes the steady state, mean square of the observation error under the assumption that the process disturbance d is Gaussian white noise with spectral density 1 and the sensor noise n is Gaussian white noise with spectral density 0.1.

(c) (5 pts) Letting $\alpha = K$ represent the state feedback gain in part (a) and $\beta = L$ the observer feedback gain in part (b), compute the controller transfer function resulting from applying the optimal state feedback gain to the estimated state. Under what conditions is the closed loop system stable?