CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 112/Ae 103a

Homework Set #8

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Problem 1. Consider a discrete-time, scalar linear system with dynamics

$$x[k+1] = ax[k] + v[k], \qquad y[k] = x[k] + w[k],$$

where v and w are discrete-time, Gaussian random processes with mean zero and variances 1 and σ^2, respectively. Assume that the initial value of the state has zero mean and variance σ_0^2.

(a) Compute the optimal estimator for the system using y as a (noisy) measurement. Your solution should be in the form of an explicit, time-varying, discrete-time system.

(b) Write down an explicit formula for the steady-state optimal estimator assuming that all transients have died out. What is the steady steady covariance of the estimator error?

(c) Suppose the mean value of the initial condition is $\mathbb{E}(x[0]) = 1$ and $a = 1$. Determine the optimal steady-state estimator for the system.

Problem 2. Consider a simple spring mass system that has two sensors: a position sensor (based on GPS) and a velocity sensor. In this problem we will use the predictor-corrector form of the discrete-time Kalman filter equations to simulate the convergence of the covariance estimate of the position and velocity of the mass.

The dynamics of the system are given by

$$m\ddot{q} + b\dot{q} + kq = v$$

where $m = 10$ g is the mass of the system, $b = 0.1$ N/m/s is the damping coefficient, and $k = 1$ N/m is the spring constant. We assume that v consists of white noise with intensity 0.1 N.

Assume that the position measurement is done at 5 Hz with an accuracy (standard deviation of each estimate) of 1 m and that the velocity measurement is done at 20 Hz with an accuracy of 0.1 m/s.

Convert the system to the equivalent discrete-time system, and compute the time history of the error covariance for the optimal estimate in predictor-corrector form. Plot the diagonal elements of $P_q[k|k]$ and $P_{\dot{q}}[k|k]$, corresponding to the accuracy of the position and velocity estimate.

(To handle the fact that the initial estimate is not known, assume that the estimator is initialized with a (noisy) measurement from the position and velocity sensors.)

Problem 3. Consider the problem of estimating the position of a car operating on a road whose dynamics are modeled as described in Example 2.3 in OBC. We assume that the car is executing a lane change manuever and we wish to estimate its position using a set of available sensors:

- A stereo camera pair, which relatively poor longitudinal (x) accuracy but good lateral position (y) accuracy. We model the covaraiance of the sensor noise as $R_{\text{lat}} = \text{diag}(1, 0.1)$.
- An automotive grade radar, which has good longitudinal position (x) accuracy but poor lateral (y) accuracy, with $R_{\text{lon}} = \text{diag}(0.1, 1)$.
- We assume the radar can also measure the longitudinal velocity ($\dot{x}$) as an optional measurement, with $R_{\text{vel}} = 1$.

In this problem we assume that the detailed model of the system is not known and also that the inputs to the vehicle (throttle and steering) are not known. We use a variety of system models to explore how these different measurements can be fused to obtain estimates and predictions of the vehicle position.

(a) Consider a model of the vehicle consisting of a particle in 2D, with the velocity of particle in the x and y direction taken as the input:

$$\dot{x} = u_1, \qquad \dot{y} = u_2$$

A discrete-time version of the system dynamics is given by

$$x[k+1] = x[k] + u_1[k] * T_{\text{s}}, \qquad y[k+1] = y[k] + u_2[k] * T_{\text{s}},$$

where $T_s = 0.1$ s is the sampling time between sensor measurements.

Construct an estimator for the system using a combination of the stereo pair and the radar (position only). Estimate the state and covariance of the system during the lane change manuever from Example 2.3 and predict the state for the next 4 seconds.

(b) Assume now that we now add (noisy) measurement of the velocity from the radar as an approximation of the input u_1. Update your Kalman filter to utilize this measurement (with no filtering), and replot the estimate and prediction for the system.

(c) To provide a better prediction, we can increase the complexity of our model so that it includes the velocity of the vehicle as a state, allowing us to model the acceleration as the input. In continuous time, this model is given by

$$\ddot{x} = u_1, \qquad \dot{y} = u_2$$

(note that we are still modeling the lateral position using a single integrator).

Convert this model to discrete time and construct an estimator for the system using a combination of the stereo pair and the radar (position and velocity). Estimate the state and covariance of the system during the lane change manuever and predict the state for the next 4 seconds.

Note: in this problem you have quite a bit of freedom in how you model the disturbances, which should model the unknown inputs to the vehicle being observed. Make sure to provide some level of justification for how you chose these disturbances.