CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 112/Ae 103a

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Homework Set #6

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Due: 16 Feb 2022

Problem 1. Consider the motion of a particle that is undergoing a random walk in one dimension (i.e., along a line). We model the position of the particle as

$$x[k+1] = x[k] + u[k],$$

where x is the position of the particle and u is a white noise processes with $\mathbb{E}(u[i]) = 0$ and $\mathbb{E}(u[i]\,u[j])R_u\delta(i-j)$. We assume that we can measure x subject to additive, zero-mean, Gaussian white noise with covariance 1. Show that the expected value of the particle as a function of k is given by

$$\mathbb{E}(x[k]) = \mathbb{E}(x[0]) + \sum_{i=0}^{k-1} \mathbb{E}(u[i]) = \mathbb{E}(x[0]) =: \mu_x$$

and the covariance $\mathbb{E}((x[k]-\mu_x)^2)$ is given by

$$\mathbb{E}((x[k]-\mu_x)^2) = \sum_{i=0}^{k-1} \mathbb{E}(u^2[i]) = kR_u$$

Problem 2. Consider a second order system with dynamics

$$\begin{bmatrix} \dot X_1 \\ \dot X_2 \end{bmatrix} = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v, \qquad Y \;= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

that is forced by Gaussian white noise with zero mean and variance σ^2. Assume $a, b > 0$.

(a) Compute the correlation function $r(\tau)$ for the output of the system. Your answer should be an explicit formula in terms of a, b and σ.

(b) Assuming that the input transients have died out, compute the mean and variance of the output.

Problem 3. Consider the dynamics of the vectored thrust aircraft described in Examples 2.4 and 3.5 with disturbances added in the x and y coordinates:

$$\begin{aligned} m\ddot x &= F_1 \cos\theta - F_2 \sin\theta - c\dot x + D_x, \\ m\ddot y &= F_1 \sin\theta + F_2 \cos\theta - c\dot y - mg + D_y, \\ J\ddot\theta &= rF_1. \end{aligned} \tag{1}$$

The measured values of the system are the position and orientation, with added noise N_x, N_y, and N_θ:

$$\vec Y = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix}. \tag{2}$$

Assume that the input disturbances are modeled by independent, first order Markov (Ornstein-Uhlenbeck) processes with $Q_D = \text{diag}(0.01, 0.01)$ and $\omega_0 = 1$ (see Definition 5.7) and that the noise is modeled as white noise with covariance matrix

$$Q_N = \begin{bmatrix} 2 \times 10^{-4} & 0 & 1 \times 10^{-5} \\ 0 & 2 \times 10^{-4} & 1 \times 10^{-5} \\ 1 \times 10^{-5} & 1 \times 10^{-5} & 1 \times 10^{-4} \end{bmatrix}$$

(a) Create disturbance and noise vectors with the desired characteristics and then compute the sample mean, covariance, and correlation, showing that they match the specifications.

(b) Create a simulation of the PVTOL system with noise and disturbance inputs and plot the response of the system from an initial equilibrium position $(x_0, y_0) = (2, 1)$ to the origin using an LQR compensator with weights

$$Q_x = \text{diag}([1, 1, 10, 0, 0, 0]), Q_u = \text{diag}([10, 1]).$$

(c) Compute the linearization of the system and find the (steady state) mean and variance of the output of the linearized system as a function of time.

(d) Use the linearization to compute the expected correlation function for the output vector $\vec{Y}$. Plot the (auto) correlation function for x, y, and θ.

(e) Using the "stationary" portion of your simulation from part (b), compute the sample mean, sample variance, and sample correlation for the output Y. Compare these to the calculations from parts (c) and (d).