CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 112/Ae 103a
Homework Set #5

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Problem 1. Consider the double integrator system from Example 4.1. A discrete time representation of this system with sampling time of 1 second is given by

$$x[k+1] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{clip}(u), \qquad \text{where} \qquad \text{clip}(u) = \begin{cases} -1 & u < -1, \\ u & -1 \leq u \leq 1, \\ 1 & u > 1. \end{cases}$$

We choose the same weighting matrices as in Example 4.1:

$$Q_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad Q_u = \begin{bmatrix} 1 \end{bmatrix}, \qquad P_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

(a) Construct a discrete-time receding horizon control law for the system and recreate Figure 4.3 using $x_0 = (2, 1)$ as the initial condition. Your plot should should show the actual trajectory for x and u as solid lines and the predicted trajectories from the optimization as dashed lines.

(b) The discrete time equivalent of the conditions in Theorem 4.1 are

$$\min_u V(f(x,u)) - V(x) + L(x,u) \leq 0 \quad \text{for all} \quad x,$$

where f represents the discrete time dynamics $x[k+1] = f(x[k], u[k])$. Check to see if these conditions are satisfied for this system using the weights above along the states that are visited along the trajectory of the system in (a). (For Theorem 4.1 to hold you would need to show this condition at all states x, so we are just checking a subset in this problem.)

(c) Replace the terminal cost P_1 with the solution to the discrete time algebraic Riccati equation (which can be obtained using the `dlqr` command in MATLAB or Python), recompute and plot the initial condition response of the receding horizon controller, and check that whether satisfies the stability condition along the states in the trajectory.

(d) Modify the terminal cost P_1 obtained in part (c) by 10X in each direction ($P_1' = 0.1 P_1$ and $P_1' = 10 P_1$), recompute and plot the initial condition response of the receding horizon controller, and check that whether satisfies the stability condition along the trajectory.

Problem 2. Consider the optimal control problem given in Example 3.2:

$$\dot{x} = ax + bu, \qquad J = \tfrac{1}{2} \int_{t_0}^{t_f} u^2(t)\, dt + \tfrac{1}{2} c x^2(t_f),$$

where $x \in \mathbb{R}$ is a scalar state, $u \in \mathbb{R}$ is the input, the initial state $x(t_0)$ is given, and $a, b \in \mathbb{R}$ are positive constants. We take the terminal time t_f as given and let $c > 0$ be a constant that balances the final value of the state with the input required to get to that position. The optimal control for a finite time $T > 0$ is derived in Example 3.2. Now consider the infinite horizon cost

$$J = \tfrac{1}{2} \int_{t_0}^{\infty} u^2(t)\, dt$$

with $x(t)$ at $t = \infty$ constrained to be zero.

(a) Solve for $u^*(t) = -bPx^*(t)$ where P is the positive solution corresponding to the algebraic Riccati equation. Note that this gives an explicit feedback law ($u = -bPx$).

(b) Plot the state solution of the finite time optimal controller for the following parameter values

$$\begin{aligned} a &= 2 \qquad b = 0.5 \qquad x(t_0) = 4 \\ c &= 0.1,\ 10 \qquad t_f = 0.5,\ 1,\ 10 \end{aligned}$$

(This should give you a total of 6 curves.) Compare these to the infinite time optimal control solution. Which finite time solution is closest to the infinite time solution? Why?

(c) Using the solution given in equation (3.5), implement the finite-time optimal controller in a receding horizon fashion with an update time of $\delta = 0.5$. Using the parameter values in part (b), compare the responses of the receding horizon controllers to the LQR controller you designed in part (a), from the same initial condition. What do you observe as c and t_f increase?

(Hint: you can write a Python script to do this by performing the following steps:

(i) set $t_0 = 0$
(ii) using the closed form solution for x^* from problem 1, plot $x(t)$, $t \in [t_0, t_f]$ and save $x_\delta = x(t_0 + \delta)$
(iii) set $x(t_0) = x_\delta$ and repeat step (ii) until x is small.)

Problem 3. Consider the dynamics of the vectored thrust aircraft described in Examples 2.4 and 3.5. The equations of motion are given by

$$\begin{aligned} m\ddot{x} &= F_1 \cos\theta - F_2 \sin\theta - c\dot{x}, \\ m\ddot{y} &= F_1 \sin\theta + F_2 \cos\theta - c\dot{y} - mg, \\ J\ddot{\theta} &= rF_1. \end{aligned} \tag{1}$$

with parameter values

$$m = 4 \text{ kg}, \qquad J = 0.0475 \text{ kg m}^2, \qquad r = 0.25 \text{ m}, \qquad g = 9.8 \text{ m/s}^2, \qquad c = 0.05 \text{ Ns/m},$$

which corresponds roughly to the values for the Caltech ducted fan flight control testbed. Assume that the inputs must satisfy the constraints

$$|F_1| \le 0.1\,|F_2|, \qquad 0 \le F_2 \le 1.5\,mg.$$

Design a receding horizon controller for the system that stabilizes the origin using an optimization horizon of $T = 5$ s and an update period of $\Delta T = 0.1$ s. Demonstrate the performance of your controller from initial conditions starting at initial position $(x_0,\ y_0) = (0 \text{ m}, 5 \text{ m})$ and desired final position $(x_\text{f},\ y_\text{f}) = (10 \text{ m}, 5 \text{ m})$ (all other states should be zero).