CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 112/Ae 103a

R. Murray | Homework Set #4 | Issued: 26 Jan 2022
Winter 2022 | | Due: 2 Feb 2022

Problem 1. Consider the double integrator

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u = Ax + Bu.$$

Find a state feedback that minimizes the quadratic cost function

$$J = \int_0^\infty \left(q_1 x_1^2 + q_2 x_2^2 + q_u u^2\right) dt$$

where $q_1 \geq 0$ is the penalty on position, $q_2 \geq 0$ is the penalty on velocity, and $q_u > 0$ is the penalty on control actions. Analyze the coefficients of the closed loop characteristic polynomial and explore how they depend on the penalties:

(a) Show that the locations of the closed loop poles depend only on the ratios of the weights q_i.

(b) Determine which of the weights has the most significant effect on the bandwidth (and hence response speed) of the closed loop system.

(c) Determine the conditions under which the closed loop poles are critically damped ($\zeta = 1/\sqrt{2}$).

Problem 2. Consider a linear system of the form

$$\frac{dx}{dt} = \begin{bmatrix}0 & 2\\-1 & -0.1\end{bmatrix}x + \begin{bmatrix}0\\1\end{bmatrix}(u + d)$$

where d is a disturbance input.

(a) Design an LQR controller for the system that regulates the system to a desired equilibrium point $x_\text{d} = (1, 0)$, assuming no disturbance ($d = 0$). Plot the response of the system starting from an initial condition $x(0) = (0, 0)$ and show that the system response converges to the desired equilibrium point.

(b) Assume now that $d = 0.1$. Show that the system response with your LQR controller from part (a) no longer converges to the desired equilibrium point and construct a controller using integral action that recovers the ability to drive the system state to that point.

(c) Suppose that there is uncertainty in the input matrix, so that the B matrix becomes

$$B = \begin{bmatrix}0\\\gamma\end{bmatrix}, \qquad 0.5 \leq \gamma \leq 2.$$

Show that the uncertain system response with your original LQR controller no longer converges to the desired equilibrium (even with $d = 0$), but that the controller with integral action still causes the system to converge to the desired equilibrium. (OK to just show this for $\gamma = 0.5$ and 2.)

Problem 3. Consider the dynamics of the vectored thrust aircraft described in Examples 2.4 and 3.5. The equations of motion are given by

$$\begin{aligned} m\ddot{x} &= F_1 \cos\theta - F_2 \sin\theta - c\dot{x}, \\ m\ddot{y} &= F_1 \sin\theta + F_2 \cos\theta - c\dot{y} - mg, \\ J\ddot{\theta} &= rF_1. \end{aligned} \tag{1}$$

with parameter values

$$m = 4\ \text{kg}, \qquad J = 0.0475\ \text{kg m}^2, \qquad r = 0.25\ \text{m}, \qquad g = 9.8\ \text{m/s}^2, \qquad c = 0.05\ \text{Ns/m},$$

which corresponds roughly to the values for the Caltech ducted fan flight control testbed.

(a) Design a feasible trajectory (x_d, u_d) for the system that corresponds to moving to the right by 10 meters over a period of 5 seconds. (For the purpose of designing this trajectory, you can assume $c = 0$ so that the system is differentially flat.) Plot the open loop response of the system (with $c \neq 0$) when the desired input u_d is applied.

(b) Design a time-invariant linear controller for the system that attempts to track the trajectory using pure feedback, of the form $u = u_\text{eq} - K(x - x_\text{d})$, where u_eq is the input required to hold the system at hover (replacing u_d). Plot the response of the system along with the inputs.

(c) Add back in the feedforward term u_d and compare the performance (errors) of the pure feedback controller ($u = u_\text{eq} - K(x - x_\text{d})$) with the controller including feedforward ($u = u_\text{d} - K(x - x_\text{d})$).

(d) Suppose that the system is subject to a disturbance force due to wind blowing from the right, so that the dynamics in the x coordinate become

$$m\ddot{x} = F_1 \cos\theta - F_2 \sin\theta - c\dot{x} + d$$

with $d = -5$. Design a linear feedback controller using integral action on the x and y errors and compare the performance of the controller with integral action to the controller with feedforward and feedback but no integral action.