CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 112/Ae 103a

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Winter 2022 | | Due: 12 Jan 2022

Problem 1. Consider a closed loop system with process dynamics and a PI controller modeled by

$$\frac{dy}{dt} + ay = bu, \qquad u = k_\mathrm{p}(r - y) + k_\mathrm{i} \int_0^t \bigl(r(\tau) - y(\tau)\bigr)\, d\tau,$$

where r is the reference, u is the control variable, and y is the process output.

(a) Derive a differential equation relating the output y to the reference r by direct manipulation of the equations and compute the transfer function $H_{yr}(s)$ (written as a ratio of two polynomials in s).

(b) Draw a block diagram of the system and derive the transfer functions of the process $P(s)$ and the controller $C(s)$.

(c) Use block diagram algebra to compute the transfer function from reference r to output y of the closed loop system and verify that your answer matches your answer in part (a).

(d) Show that if the closed loop system is stable, the steady state error converges to zero.

Problem 2. Consider a control system with

$$P(s) = \frac{b}{(s + a)^2}, \qquad C(s) = \frac{k_\mathrm{p} s + k_\mathrm{i}}{s},$$

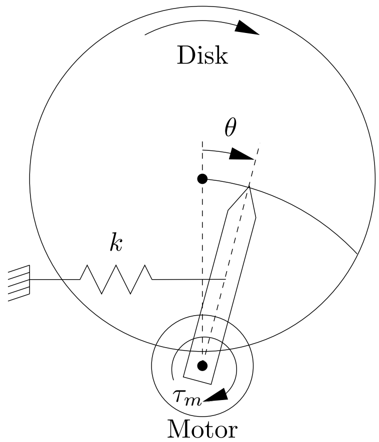
and set $a = b = 1$ and $k_\mathrm{p} = 1$, $k_\mathrm{i} = 0.1$. Using the Python Control Systems Library (python-control), do the following:

(a) Plot the step response of the closed loop system and compute the rise time, settling time, and steady state error.

(b) Plot the frequency response of the open loop system (Bode plot) and compute the gain margin, phase margins, and bandwidth of the system.

(c) Plot the Nyquist plot of the system and compute the stability margin (smallest distance to the -1 point).

Problem 3. Consider a simple mechanism for positioning a mechanical arm and the associated equations of motion:



$$J\ddot{\theta} = -b\dot{\theta} - kr\sin\theta + \tau_m$$
$$\dot{\tau}_m = -a(\tau_m - u)$$

The system consists of a spring-loaded arm that is driven by a motor. The motor applies a force against the spring and pulls the tip across a rotating platter. The input to the system is the desired motor torque, u. In the diagram above, the force exerted by the spring is a nonlinear function of the head position due to the way it is attached. Take the system parameters to be

$$k = 1, \quad J = 100, \quad b = 10, \quad r = 1, \quad l = 2, \quad \epsilon = 0.01.$$

Starting with the template Jupyter notebook posted on the course website, create a Jupyter notebook that documents the following operations:

(a) Compute the linearization of the dynamics about the equilibrium point corresponding to $\theta_e = 15°$.

(b) Plot the step response of the linearized, open-loop system and compute the rise time and settling time.

(c) Design a state feedback controller for the system that stabilizes the system about $\theta_e = 45°$ and sets the closed loop eigenvalues to $\lambda_{1,2} = -1 \pm \sqrt{3}i$. Plot the step response for the closed loop system and compute the rise time, settling time, and steady state error.

(d) Compute the transfer function H_{yu} for the open system around the equilibrium point and sketch the frequency response of the open loop system.

(e) Design a frequency domain compensator that provides tracking with less than 10% error up to 1 rad/sec and has a phase margin of at least 45°. Demonstrate that your controller meets these requirements by showing Bode, Nyquist, and step response plots, and compute the rise time, settling time, and steady state error for the system using your controller design.

(f) Create simulations of the full nonlinear system with the linear controllers designed in parts (c) and (e) and plot the response of the system from an initial position of 0 mm at $t = 0$, to 0.4 mm at $t = 30$ ms, to 1.2 mm at $t = 90$ ms, to 0.8 mm at $t = 120$ ms.