CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b

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Note: Please put the number of hours that you spent on this homework set (including reading) on the back of the first page of your homework.

1. Consider the following linear system

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u + \begin{bmatrix}1\\1\end{bmatrix}v$$
$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + w$$

where v and w are Gaussian white noise with covariances $r > 0$ and 1, respectively. Suppose we wish to design a controller that minimizes the cost function

$$J = \int_0^\infty qx^T \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} x + u^2 dt$$

where $q > 0$.

   (a) Design a controller for the system using a Kalman filter and optimal linear quadratic regulator. Give the transfer function for the resulting compensator.

   (b) Show that the resulting closed loop system has vanishingly small gain margin for r and q chosen sufficiently large. (Hint: you should spend about 30 minutes trying this problem and then go and read the 1976 paper by Doyle [available on the course web page]. Be careful about differences in notation between the paper and the problem statement.)

2. Consider the class of perturbed plants of the form

$$\tilde{P} = \frac{P}{1 + \delta_{\text{fb}} W_2 P},$$

where W_2 is a fixed stable weighting function with W_2 strictly proper and $\delta_{\text{fb}}(s)$ is an unknown stable transfer function with $\|\delta_{\text{fb}}\|_\infty \leq 1$. Assume that C is a controller achieving stability for P. Prove that C provides internal stability for the perturbed plant if $\|W_2 P S\|_\infty < 1$.

Students who are not doing the course project should complete the following problem (worth 20 points):

3. This problem shows that the stability margin is critically dependent on the type of perturbation. The setup is a unity-feedback loop with controller $C(s) = 1$ and process dynamics $\tilde{P}(s) = P(s) + \Delta(s)$, where

$$P(s) = \frac{10}{s^2 + 0.2s + 1}$$

(a) Assume $\Delta(s)$ is a stable transfer function. Compute the largest β such that the feedback system is internally stable for all $\|\Delta\|_\infty < \beta$.

(b) Now suppose that Δ is a real number. Determine the bounds on Δ such that the closed loop system is stable and compare to the first part. (Hint: compute the closed loop transfer function analytically and determine when the eigenvalues go unstable.)

4. Consider the following model for the pitch dynamics of the Caltech ducted fan:

$$P(s) = \frac{r}{Js^2 + bs + mgl} \qquad \begin{array}{lll} g = 9.8 \text{ m/sec}^2 & m = 1.5 \text{ kg} & b = 0.05 \text{ kg/sec} \\ l = 0.05 \text{ m} & J = 0.0475 \text{ kg m}^2 & r = 0.25 \text{ m} \end{array}$$

We wish to design a robust controller that satisfies the following: performance specification:

- Steady state error of less than 1%
- Tracking error of less than 5% from 0 to 1 Hz (remember to convert this to rad/sec).

(a) Write the above specification as a weighted sensitivity specification. Choose an explicit weight $W_1(s)$ so that the specification is satisfied if $\|W_1 S\| < 1$ and show that the specification can be satisfied using a proportional controller.

(b) Consider a plant perturbation of 20% variation in the value of r around the nominal value. Design a controller that satisfies the nominal specification and provides robust stabilty with respect to this perturbation. (Hint: You can use any technique to design this controller, but you might try designing an estimator + state feedback controller as a first cut to see if you can do it. If you can't find a controller that satisfies the specification after a couple of attempts, try a lead compensator.)