3

OPTIMAL CONTROL OF CONTINUOUS-TIME SYSTEMS

We shall now discuss optimal control for systems with a continuous time index. From a glance at the table of contents, it is apparent that this chapter will follow the development of Chapter 2.

There are several distinctions between the optimal control problems for continuous and discrete systems, the most noticeable of which is that the continuous control laws are based on equations of a simpler form than their discrete counterparts. That will allow us to obtain some analytic solutions in this chapter.

Another distinction arises in the initial stages of the derivation of the control law. For continuous systems, we must distinguish between *differentials* and *variations* in a quantity, which we did not need to do in Chapter 2. This means that we shall need to use the calculus of variations, which is briefly reviewed in Section 3.1.

The continuous dependence on time also makes it fairly simple to talk about minimum-time problems, which we do in Chapter 5.

The derivations in this chapter are for the most part similar to those for discrete systems, and we shall attempt to set them down in a manner that makes clear what is going on without duplicating too much of our work from Chapter 2.

3.1 THE CALCULUS OF VARIATIONS

Only a few ideas from the calculus of variations will be needed, so our review will be short. For an in-depth discussion, see Athans and Falb (1966) or Kirk (1970).

In Section 3.2 we shall be concerned with minimizing an augmented performance index J' exactly as we were in Chapter 2. To perform this minimization, we shall need to find the change induced in J' by independent changes in all of its arguments (cf. (2.1-6)).

Unfortunately, we shall run into a slight problem. The change in J' will depend on the time and state differentials dt and dx. However, these quantities are not independent. The purpose of this section is to clear up this point and to derive a relation that will soon be useful.

If $x(t)$ is a continuous function of time t, then the differentials $dx(t)$ and dt are not independent. We can, however, define a small change in $x(t)$ that *is* independent of dt. Let us define the *variation* in $x(t)$, $\delta x(t)$, as the incremental change in $x(t)$ when time t is held fixed.

To find the relations among dx, δx, and dt, examine Fig. 3.1-1. Here we show the original function $x(t)$ and a neighboring function $x(t) + dx(t)$ over an interval specified by initial time t_0 and final time T (Bryson and Ho 1975). In addition to the increment $dx(t)$ at each time t, the final time has been incremented by dT. It is clear from the illustration that the overall increment in x at T, $dx(T)$, depends on dT. According to our definition, the variation $\delta x(T)$ occurs at the fixed value of $t = T$ as shown and is independent of dT. Since $x(t)$ and $x(t) + dx(t)$ have approximately the same slope $\dot{x}(T)$ at $t = T$, and since dT is small, we have

$$dx(T) = \delta x(T) + \dot{x}(T)\, dT. \tag{3.1-1}$$

This relation is the one we shall need later.

Another relation we shall need is *Leibniz's rule* for functionals: if $x(t) \in R^n$ is a function of t and

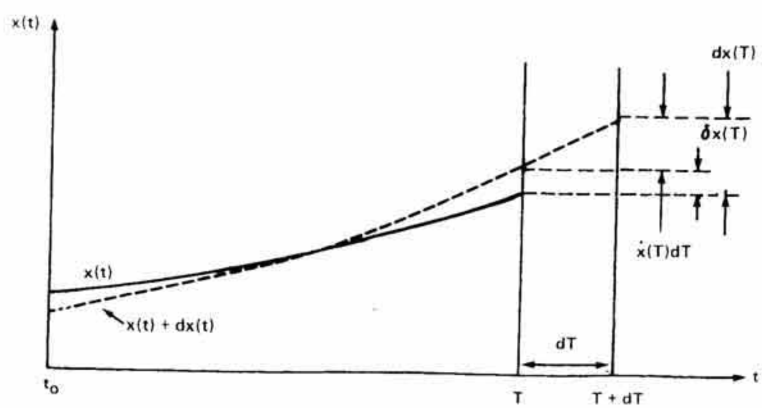

FIGURE 3.1-1 Relation between the variation δx and the differential dx.

$$J(x) = \int_{t_0}^{T} h(x(t), t)\, dt, \tag{3.1-2}$$

where $J(\cdot)$ and $h(\cdot)$ are both real scalar functionals (i.e., functions of the function $x(t)$), then

$$dJ = h(x(T), T)\, dT - h(x(t_0), t_0)\, dt_0 + \int_{t_0}^{T} [h_x^T(x(t), t)\delta x]\, dt. \tag{3.1-3}$$

Our notation is

$$h_x \triangleq \frac{\partial h}{\partial x}.$$

3.2 SOLUTION OF THE GENERAL CONTINUOUS OPTIMIZATION PROBLEM

The philosophy in this chapter is to derive the solution to the continuous optimal control problem in the most general case. This is accomplished in the present section. Then, in subsequent sections, we consider various special cases of the general solution.

The discussion at the beginning of Chapter 2 and the comments in Section 2.1 also apply here; they provide some insight on the formulation of the optimal control problem.

Problem Formulation and Solution

Suppose the plant is described by the nonlinear time-varying dynamical equation

$$\dot{x}(t) = f(x, u, t), \tag{3.2-1}$$

with state $x(t) \in R^n$ and control input $u(t) \in R^m$. With this system let us associate the performance index

$$J(t_0) = \phi(x(T), T) + \int_{t_0}^{T} L(x(t), u(t), t)\, dt, \tag{3.2-2}$$

where $[t_0, T]$ is the time interval of interest. The final weighting function $\phi(x(T), T)$ depends on the final state and final time, and the weighting function $L(x, u, t)$ depends on the state and input at intermediate times in $[t_0, T]$.

The performance index is selected to make the plant exhibit a desired type of performance. Some different possibilities for $J(t_0)$ are discussed in Example 2.1-1, which carries over to the continuous case.

The *optimal control problem* is to find the input $u^*(t)$ on the time interval $[t_0, T]$ that drives the plant (3.2-1) along a trajectory $x^*(t)$ such that the cost function (3.2-2) is minimized, *and* such that

$$\psi(x(T), T) = 0 \tag{3.2-3}$$

for a given function $\psi \in R^p$.

This corresponds to the function-of-final-state-fixed discrete problem solved in Section 4.5.

The roles of the final weighting function ϕ and the fixed final function ψ should not be confused. $\phi(x(T), T)$ is a function of the final state, which we want to make *small*. An illustration might be the energy, which is $[x^T(T)S(T)x(T)]/2$, where $S(T)$ is a given weighting matrix. On the other hand, $\psi(x(T), T)$ is a function of the final state, which we want *fixed* at exactly zero. As an illustration, consider a satellite with state $x = [r\ \dot{r}\ \theta\ \dot{\theta}]^T$, where r and θ are radius and angular position. If we want to place the satellite in a circular orbit with radius R, then the final state function to be zeroed would be

$$\psi(x(T), T) = \begin{bmatrix} r(T) - R \\ \dot{r}(T) \\ \dot{\theta}(T) - \sqrt{\dfrac{\mu}{R^3}} \end{bmatrix},$$

with $\mu = GM$ the gravitational constant of the attracting mass M.

To solve the continuous optimal control problem, we shall use Lagrange multipliers to adjoin the constraints (3.2-1) and (3.2-3) to the performance index (3.2-2). Since (3.2-1) holds at each $t \in [t_0, T]$, we require an associated multiplier $\lambda(t) \in R^n$, which is a function of time. Since (3.2-3) holds only at one time, we require only a constant associated multiplier $\nu \in R^p$. The augmented performance index is thus

$$J' = \phi(x(T), T) + \nu^T\psi(x(T), T) + \int_{t_0}^{T} [L(x, u, t) + \lambda^T(t)(f(t)(x, u, t) - \dot{x})]\, dt. \tag{3.2-4}$$

If we define the *Hamiltonian function* as

$$H(x, u, t) = L(x, u, t) + \lambda^T f(x, u, t), \tag{3.2-5}$$

then we can rewrite (3.2-4) as

$$J' = \phi(x(T), T) + \nu^T\psi(x(T), T) + \int_{t_0}^{T} [H(x, u, t) - \lambda^T\dot{x}]\, dt. \tag{3.2-6}$$

Using Leibniz's rule, the increment in J' as a function of increments in x, λ, ν, u, and t is

$$\begin{aligned} dJ' = {} & (\phi_x + \psi_x^T\nu)^T\, dx|_T + (\phi_t + \psi_t^T\nu)\, dt|_T + \psi^T|_T\, d\nu \\ & + (H - \lambda^T\dot{x})\, dt|_T - (H - \lambda^T\dot{x})\, dt|_{t_0} \\ & + \int_{t_0}^{T} [H_x^T\, \delta x + H_u^T\, \delta u - \lambda^T\delta\dot{x} + (H_\lambda - \dot{x})^T\, \delta\lambda]\, dt. \end{aligned} \tag{3.2-7}$$

To eliminate the variation in $\dot{x}$, integrate by parts to see that

$$-\int_{t_0}^{T} \lambda^T\, \delta\dot{x}\, dt = -\lambda^T\, \delta x|_T + \lambda^T\, \delta x|_{t_0} + \int_{t_0}^{T} \dot{\lambda}^T\, \delta x\, dt. \tag{3.2-8}$$

If we substitute this into (3.2-7), there result terms at $t = T$ dependent on both $dx(t)$ and $\delta x(T)$. We can express $\delta x(T)$ in terms of $dx(t)$ and dT using (3.1-1). The result after these two substitutions is

$$\begin{aligned} dJ' = {} & (\phi_x + \psi_x^T\nu - \lambda)^T\, dx|_T + (\phi_t + \psi_t^T\nu + H - \lambda^T\dot{x} + \lambda^T\dot{x})\, dt|_T \\ & + \psi^T|_T\, d\nu - (H - \lambda^T\dot{x} + \lambda^T\dot{x})\, dt|_{t_0} + \lambda^T\, dx|_{t_0} \\ & + \int_{t_0}^{T} [(H_x + \dot{\lambda})^T\, \delta x + H_u^T\, \delta_u + (H_\lambda - \dot{x})^T\, \delta\lambda]\, dt. \end{aligned} \tag{3.2-9}$$

According to the Lagrange theory, the constrained minimum of J is attained at the unconstrained minimum of J'. This is achieved when $dJ' = 0$ for all independent increments in its arguments. Setting to zero the coefficients of the independent increments $d\nu$, δx, δu, and $\delta\lambda$ yields necessary conditions for a minimum as shown in Table 3.2-1. For our applications, t_0 and $x(t_0)$ are both fixed and known, so that dt_0 and $dx(t_0)$ are both zero. The two terms evaluated at $t = t_0$ in (3.2-9) are thus automatically equal to zero.

The final condition (3.2-10) in the table needs further discussion. We have seen that $dx(T)$ and dT are not independent (Fig. 3.1-1). Therefore, we cannot

TABLE 3.2-1 Continuous Nonlinear Optimal Controller with Function of Final State Fixed

System model:

$$\dot{x} = f(x, u, t), \qquad t \geq t_0, \qquad t_0 \text{ fixed}$$

Performance index:

$$J(t_0) = \phi(x(T), T) + \int_{t_0}^{T} L(x, u, t)\, dt$$

Final-state constraint:

$$\psi(x(T), T) = 0$$

Optimal controller:
Hamiltonian:

$$H(x, u, t) = L(x, u, t) + \lambda^T f(x, u, t)$$

State equation:

$$\dot{x} = \frac{\partial H}{\partial \lambda} = f, \qquad t \geq t_0$$

Costate equation:

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = \frac{\partial f^T}{\partial x}\lambda + \frac{\partial L}{\partial x}, \qquad t \leq T$$

Stationarity condition:

$$0 = \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \frac{\partial f^T}{\partial u}\lambda$$

Boundary condition:

$$x(t_0) \text{ given}$$

$$(\phi_x + \psi_x^T \nu - \lambda)^T|_T\, dx(T) + (\phi_t + \psi_t^T \nu + H)|_T\, dT = 0 \qquad (3.2\text{-}10)$$

simply set the coefficients of the first two terms on the right-hand side of (3.2-9) separately equal to zero. Instead, the entire expression (3.2-10) must be zero at $t = T$. Compare it with (2.7-7). The extra term in (3.2-10) arises in the present situation since we have allowed for possible variations in the final time T. This will allow us to deal with minimum-time problems, which we shall do in Chapter 5.

For convenience, we have shown the conditions in the table both in terms of H and in terms of L and f. Compare these results with Table 2.1-1, and see the associated discussion for further insight. Note that the discrete and continuous costate equations are both dynamical equations that develop *backward* in time. In the continuous case, this amounts to making the rate of change (i.e., $\dot{\lambda}$) negative. The costate equation is also called the *adjoint* to the state equation.

As in the discrete case, the optimal control in Table 3.2-1 depends on the solution to a two-point boundary-value problem, since $x(t_0)$ is given and $\lambda(T)$ is determined by (3.2-10). It is in general very difficult to solve these problems.

We do not really care about the value of $\lambda(t)$, but it must evidently be determined as an intermediate step in finding the optimal control $u^*(t)$, which depends on $\lambda(t)$ through the stationarity condition.

An important point is worth noting. The time derivative of the Hamiltonian is

$$\dot{H} = H_t + H_x^T \dot{x} + H_u^T \dot{u} + \dot{\lambda}^T f = H_t + H_u^T \dot{u} + (H_x + \dot{\lambda})^T f. \qquad (3.2\text{-}11)$$

If $u(t)$ is an optimal control, then

$$\dot{H} = H_t. \qquad (3.2\text{-}12)$$

Now, in the time-invariant case, f and L are not explicit functions of t, and so neither is H. In this situation

$$\dot{H} = 0. \qquad (3.2\text{-}13)$$

Hence, for time-invariant systems and cost functions, the Hamiltonian is a *constant* on the optimal trajectory.

Let us begin to develop a feel for the continuous optimal controller by looking at some examples.

Some Examples

The first two examples make the point that the solution to the optimization problem given in Table 3.2-1 is very general; it does not only apply in system theory. The next examples illustrate the computation of the optimal controller

$$x(a) = A, \tag{2}$$

$$x(b) = B. \tag{3}$$

See Shultz and Melsa (1967).

It is desired to find the curve $x(t)$ joining (a, A) and (b, B) that minimizes (1).

To put this into the form of an optimal control problem, define the "input" by

$$\dot{x} = u. \tag{4}$$

This is the "plant." Then (1) becomes

$$J = \int_a^b \sqrt{1 + u^2}\, dt. \tag{5}$$

The Hamiltonian is

$$H = \sqrt{1 + u^2} + \lambda u \tag{6}$$

Now, Table 3.2-1 yields the conditions

$$\dot{x} = H_\lambda = u, \tag{7}$$

$$-\dot{\lambda} = H_x = 0, \tag{8}$$

$$0 = H_u = \lambda + \frac{u}{\sqrt{1 + u^2}}. \tag{9}$$

To solve these for the optimal slope u, note that by (9)

$$u = \frac{-\lambda}{\sqrt{1 - \lambda^2}}, \tag{10}$$

but according to (8), λ is constant. Hence,

$$u = \text{const} \tag{11}$$

is the optimal "control." Now use (7) to get

$$x(t) = c_1 t + c_2. \tag{12}$$

To determine c_1 and c_2, use the boundary conditions (2) and (3) to see that

$$x(t) = \frac{(A - B)t + (aB - bA)}{a - b}. \tag{13}$$

The optimal trajectory (13) between two points is thus a straight line. ■

Example 3.2-3: Temperature Control in a Room

It is desired to heat a room using the least possible energy. If $\theta(t)$ is the temperature in the room, θ_a the ambient air temperature outside (a constant), and $u(t)$ the rate of heat supply to the room, then the dynamics are

$$\dot{\theta} = -a(\theta - \theta_a) + bu \tag{1}$$

for some constants a and b, which depend on the room insulation and so on. By defining the state as

$$x(t) \triangleq \theta(t) - \theta_a, \tag{2}$$

we can write the state equation

$$\dot{x} = -ax + bu. \tag{3}$$

See McClamroch (1980). In order to control the temperature on the fixed time interval $[0, T]$ with the least possible supplied energy, define the performance index as

$$J(0) = \frac{1}{2}\int_0^T u^2(t)\, dt. \tag{4}$$

We shall discuss two possible control objectives in parts a and b below. The Hamiltonian is

$$H = \frac{u^2}{2} + \lambda(-ax + bu). \tag{5}$$

According to Table 3.2-1, the optimal control $u(t)$ is determined by solving

$$\dot{x} = H_\lambda = -ax + bu, \tag{6}$$

$$\dot{\lambda} = -H_x = a\lambda, \tag{7}$$

$$0 = H_u = u + b\lambda. \tag{8}$$

The stationarity condition (8) says that the optimal control is given by

$$u(t) = -b\lambda(t), \tag{9}$$

so to determine $u^*(t)$ we need only find the optimal costate $\lambda^*(t)$.

Substituting (9) into (6) yields the state-costate equations

$$\dot{x} = -ax - b^2\lambda, \tag{10a}$$
$$\dot{\lambda} = a\lambda, \tag{10b}$$

which must not be solved for $\lambda^*(t)$ and the optimal state trajectory $x^*(t)$.

We do not yet know the final costate $\lambda(T)$, but let us solve (10) as if we did. The solution to (10b) is

$$\lambda(t) = e^{-a(T-t)}\lambda(T). \tag{11}$$

Using this in (10a) yields

$$\dot{x} = -ax - b^2\lambda(T)e^{-a(T-t)}. \tag{12}$$

Using Laplace transforms to solve this gives

$$\begin{aligned} X(s) &= \frac{x(0)}{s+a} - \frac{b^2\lambda(T)e^{-aT}}{(s+a)(s-a)} \\ &= \frac{x(0)}{s+a} - \frac{b^2}{a}\lambda(T)e^{-aT}\left(\frac{-1/2}{s+a} + \frac{1/2}{s-a}\right) \end{aligned} \tag{13}$$

so that

$$x(t) = x(0)e^{-at} - \frac{b^2}{a}\lambda(T)e^{-aT}\sinh at. \tag{14}$$

Equations (11) and (14) give the optimal costate $\lambda^*(t)$ and state $x^*(t)$ in terms of the as yet unknown final costate $\lambda(T)$. The initial state $x(0)$ is given.

Now we consider two possible control objectives, which will give two ways to determine $\lambda(T)$.

a. Fixed Final State

Suppose that the initial temperature of the room is equal to $\theta_a = 60°$. Then

$$x(0) = 0°. \tag{15}$$

Let our control objective be to drive the final temperature $\theta(T)$ exactly to 70° at the given final time of T seconds. Then the final state is required to take on the fixed value of

$$x(T) = 10°. \tag{16}$$

Note that since the final time and final state are both fixed, dT and $dx(T)$ are both zero, so that (3.2-10) is satisfied.

Using (15) and (16), we must determine $\lambda(T)$; then we can find $\lambda(t)$ by using (11) and the optimal control by using (9). To find $\lambda(T)$, use (14) to write

$$x(T) = x(0)e^{-aT} - \frac{b^2}{2a}\lambda(T)(1 - e^{-2aT}). \tag{17}$$

Taking into account (15) and (16) shows that the final costate is

$$\lambda(T) = \frac{20a}{b^2(1 - e^{-2aT})}, \tag{18}$$

and so the optimal costate trajectory is

$$\lambda^*(t) = -\frac{10\, ae^{at}}{b^2 \sinh aT}. \tag{19}$$

Finally, the optimal rate of heat supply to the room is given by (9) or

$$u^*(t) = \frac{10\, ae^{at}}{b \sinh aT} \qquad 0 \le t \le T. \tag{20}$$

In order to check our answer, apply $u^*(t)$ to the system (3). Solving for the state trajectory yields

$$x^*(t) = 10\frac{\sinh at}{\sinh aT}. \tag{21}$$

Indeed $x^*(T) = 10$ as desired.

b. Free Final State

Now suppose that we are not so concerned that the final state $x(T)$ be exactly 10°. Let us demand only that the control $u(t)$ minimize

$$J(0) = \tfrac{1}{2}s(x(T) - 10)^2 + \frac{1}{2}\int_0^T u^2(t)\,dt \tag{22}$$

for some weighting s (i.e., some real number s) to be selected later. If s is large, then the optimal solution will have $x(T)$ near 10°, since only then will the first term make a small contribution to the cost.

According to Table 3.2-1, the state and costate equations are still given by (10), and the optimal control by (9). Therefore, (11) and (14) are still valid.

The initial condition is still (15), but the final condition must be determined by using (3.2-10). The final time T is fixed, so $dT = 0$ and the second term of (3.2-10) is automatically equal to zero. Since $x(T)$ is not fixed, $dx(T)$ is not zero (as it was in part a). Therefore, it is required that

$$\lambda(T) = \left.\frac{\partial\phi}{\partial x}\right|_T = s(x(T) - 10). \tag{23}$$