CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 110/ChE 105

Problem Set #9

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Problem 1. Consider a second-order process of the form

$$P(s) = \frac{k}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \qquad k, \zeta, \omega_0 > 0.$$

(a) Suppose that we want the closed loop dynamics of the system to have a characteristic polynomial given by

$$p(s) = s^2 + a_1 s + a_2.$$

Compute a formula for the controller parameters of a proportional-derivative (PD) controller (k_p and k_d) that gives the desired closed loop response.

(b) Suppose that we want the closed loop dynamics of the system to have a characteristic polynomial given by

$$p(s) = s^3 + a_1 s^2 + a_2 s + a_3.$$

Compute a formula for the controller parameters of a proportional-integral-derivative (PID) controller (k_p, k_i, and k_d) that gives the desired closed loop response.

Problem 2. In this problem we will use a variety of methods to design proportional (P), proportional-integral (PI), proportional-derivative (PD), and proportional-integral-derivative (PID) controllers for a vectored thrust aircraft (see Example 3.12 in FBS2e for a description).

Use the following transfer function to represent the dynamics from the lateral input to the roll angle of the aircraft:

$$P(s) = \frac{r}{Js^2 + cs + mgl}$$

$g = 9.8$ m/s^2	$m = 1.5$ kg	$c = 0.5$ kg/s
$l = 0.05$ m	$J = 0.0475$ kg m^2	$r = 0.25$ m

(these parameters roughly correspond to the laboratory-scale experiment that we used to have a Caltech, with some extra damping to make things a bit easier).

We wish to design a feedback controller that tracks a given reference input with the following specifications:

- steady-state error of less than 1%;
- frequency response bandwidth of 10 rad/sec;
- step response overshoot of less than 20% (can show this corresponds to a phase margin of approximately 45°).

(a) Use the calculations in Tables 7.1 and 7.2 of FBS2e to find values for the closed loop poles of a second order system that meets the frequency response and overshoot requirements.

(b) Design a PD controller that sets the poles of the closed loop system to $-4 \pm -7j$. Compute the frequency response and the step response and evaluate the steady state error, bandwidth, and overshoot. Does the system satisfy the specification?

(c) Plot the step response of the open loop process dynamics ($P(s)$) and use the (original) Ziegler-Nichols tuning rules to design P, PI, and PID controllers for the system. Plot the closed loop step and frequency response for each of these controllers and determine which satisfies the specifications.

(d) Use "loop shaping" to design a PID controller that satisfies the specifications. You can do this by using the following steps to create an initial set of gains:

i. Choose the time constant for the derivative term to be centered at the desired bandwidth of the system (this will give a 45° phase lead at that frequency).

ii. Choose the time constant for the integral term to be an order of magnitude away from the zero created by the derivative term (this will make sure that the phase of the first zero and the phase of the second zero do not overlap).

iii. Choose the proportional gain of the system such that the loop transfer function crosses 1 at the desired bandwidth.

The procedure above gives you a first cut at PID gains, but you may need to adjust them to satisfy the constraints on the loop transfer function.

Plot the closed loop step and the *open loop* frequency response ($L(s)$) for your final controller and show that it satisfies the specifications.

(e) Suppose that we now impose magnitude limits on the inputs such that $|u| \leq 5$. Using the PID gains from part 2d, create a PID controller with anti-windup protection and show the resulting unit step response with and without anti-windup compensation. Explain the results in terms of the value of the integral component of the controller.

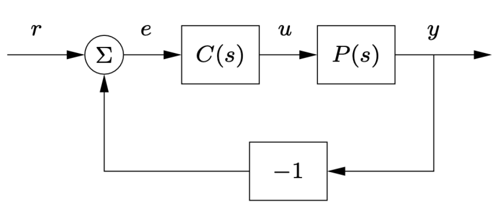
A Colab notebook template for this problem is available here.

The following problem is a conceptual problem that you should be able to do without using the textbook, your notes, or the Internet. It is representative of the type of problem you can expect on the (closed-book) final exam (so you might want to first attempt solving it without using any notes).

Problem 3. Consider a PID controller with transfer function

$$C(s) = k_\mathrm{p} + k_\mathrm{d}\frac{as}{s+a} + k_\mathrm{i}\frac{1}{s}.$$

We us this controller to implement a unity feedback control system as shown below:



Assume that the gains of the PID controller have been designed such that the closed loop system is stable (at the origin).

Which of the following statements are true? (You do not need to justify your answer.)

(a) The PID controller can be implemented as a linear system with one internal state.

(b) The PID controller is asymptotically stable at the origin.

(c) The PID controller is stable at the origin.

(d) The response of the PID controller to a constant input e is bounded.

(e) If the closed loop system with a proportional-derivative (PD) controller is stable, the output $y(t)$ for a constant reference input r always converges to r.

(f) If the closed loop system with a proportional-integral (PI) controller is stable, the output $y(t)$ for a constant reference input r always converges to r.

(g) If the closed loop system with a proportional-integral-derivative (PID) controller is stable, the output $y(t)$ for a constant reference input r always converges to r.

(h) Suppose $P(s)$ is replaced by $P'(s)$ but the controller is left fixed. If the closed loop system with the original PID controller $C(s)$ and the new process dynamics $P'(s)$ is stable, the output $y(t)$ for a constant reference input r always converges to r.