CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 110/ChE 105

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This problem set will focus on the dynamics of the vectored thrust aircraft described in Examples 2.4 and 3.5. The equations of motion are given by

$$\begin{aligned} m\ddot{x} &= F_1 \cos\theta - F_2 \sin\theta - c\dot{x}, \\ m\ddot{y} &= F_1 \sin\theta + F_2 \cos\theta - c\dot{y} - mg, \\ J\ddot{\theta} &= rF_1. \end{aligned} \tag{S6.1}$$

with parameter values

$$m = 4 \text{ kg}, \quad J = 0.0475 \text{ kg m}^2, \quad r = 0.25 \text{ m}, \quad g = 9.8 \text{ m/s}^2, \quad c = 0.05 \text{ Ns/m},$$

which corresponds roughly to the values for the Caltech ducted fan flight control testbed.

In each of the problems below, we will look at different aspects of feedforward and feedback control of this system.

Problem 1. We wish to generate an optimal trajectory for the system that corresponds to moving the system for an initial hovering position to a hovering position one meter to the right ($x_\text{f} = x_0 + 1$).

For each of the parts below, you should solve for the optimal input and plot the xy trajectory of the system, along with the angle θ and inputs F_1 and F_2 over the time interval. In addition, report the following information for each approach:

- the computation time required;
- the final position for the computed trajectory;
- the weighted integrated cost of the input along the trajectory

$$\int_0^T \big(10F_1^2(\tau) + (F_2 - mg)^2(\tau)\big)\, d\tau. \tag{S6.2}$$

  (Note that this cost may be different than the cost function that you use to determine the trajectory.)

A Jupyter notebook template is available here that can be used to answer the questions below.

(a) Solve for an optimal trajectory using a quadratic cost from the final point with weights

$$Q_x = \text{diag}([1, 1, 10, 0, 0, 0]), \qquad Q_u = \text{diag}([10, 1]).$$

This cost function attempts to minimize the angular deviation θ and the sideways force F_1.

(b) Re-solve the problem using polynomial curves as the basis functions for the inputs. This should give you smoother inputs and a nicer response.

(c) Re-solve the problem using a terminal cost $V(x(T)) = x(T)^{\mathsf{T}} P_1 x(T)$ to try to get the system closer to the final value. You should try adjusting the cost along the trajectory Q_x versus the terminal cost P_1 to minimize the weighted integrated cost (S6.2).

(d) Re-solve the problem using a terminal *constraint* to try to get the system closer to the final value. Adjust the cost along the trajectory to try to minimize the cost in equation (S6.2). (Hint: you may have to use an initial guess to get the optimization to converge.)

Problem 2. In this problem we will track a feasible trajectory $(x_{\mathrm{d}}, u_{\mathrm{d}})$ for the system that corresponds to moving to the right by 10 meters over a period of 10 seconds.

For the simulations below, you should run all simulations from an initial condition $x_{\mathrm{i}} = (0.1, -0.1, 0, 0, 0, 0)$.

A Jupyter notebook template is available here that can be used to answer the questions below.

(a) Design a time-invariant linear controller for the system that attempts to track the trajectory using pure feedback, of the form $u = u_{\mathrm{eq}} - K(x - x_{\mathrm{d}})$, where u_{eq} is the input required to hold the system at hover (replacing u_{d}). Plot the response of the system along with the inputs.

(b) Add back in the feedforward term u_{d} and compare the performance (errors) of the pure feedback controller ($u = u_{\mathrm{eq}} - K(x - x_{\mathrm{d}})$) with the controller including feedforward ($u = u_{\mathrm{d}} - K(x - x_{\mathrm{d}})$).

(c) Suppose that the system is subject to a disturbance force due to wind blowing from the right, so that the dynamics in the x coordinate become

$$m\ddot{x} = F_1 \cos\theta - F_2 \sin\theta - c\dot{x} + d$$

with $d = -6$. Plot the response of the controller from part (b) with this disturbance input.

(d) Design a gain scheduled controller that uses the roll angle θ and sidewards velocity $\dot{x}$ as gain scheduling parameters, and show that you can improve the performance of the system.

Problem 3. Assume that the inputs must satisfy the constraints

$$|F_1| \leq 1, \qquad 0 \leq F_2 \leq 50.$$

In this problem we will implement a receding horizon controller that obeys the constraints and stabilizes the step response of the system.

A Jupyter notebook template is available here that can be used to answer the questions below.

(a) Implement a simple LQR controller of the form $u = u_{\mathrm{eq}} - K(x - x_{\mathrm{f}})$ and attempt to find a cost function that converges quickly to the desired final point without going unstable.

(b) Design a receding horizon controller for the system that stabilizes the origin using an optimization horizon of $T = 3$ s and an update period of $\Delta T = 1$ s. Demonstrate the performance of your controller from initial conditions starting at initial position $(x_0,\ y_0) = (0\ \text{m},\ 5\ \text{m})$ and desired final position $(x_{\mathrm{f}},\ y_{\mathrm{f}}) = (10\ \text{m},\ 5\ \text{m})$ (all other states should be zero).