CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 110/ChE 105

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Problem 1. Consider the normalized, linearized inverted pendulum which is described by

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Ax + Bu, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Cx$$

(a) Determine the stability of the open loop system, and provide your reasoning.

(b) Compute the reachability matrix W_r and justify why the closed loop system can be made asymptotically stable.

We aim to create a stable closed loop system with the characteristic polynomial $\lambda(s) = s^2 + 2\zeta_0\omega_0 s + \omega_0^2$ and with unit static gain (steady-state output $y = r$). We will achieve this by applying a control law $u = -Kx + k_\text{f} r$, consisting of state feedback matrix $K = [k_1, k_2]$ and feedforward gain k_f.

(c) In order to do so, first determine the characteristic polynomial of the closed loop system and find the values of k_1 and k_2 that give the desired characteristic polynomial.

(d) Next, compute the equilibrium point x_e and the steady-state output y_e to determine the value of k_f that gives a closed loop system with unit static gain.

(e) Determine the values of ζ_0 and ω_0 for which the closed loop system is asymptotically stable.

Problem 2. Consider a linear system of the form

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 2 \\ -1 & -0.1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (u + d)$$

where d is a disturbance input.

(a) Create a linear state space model and plot the open loop step response.

In the next part of the problem, we'll explore a few different choices for the LQR weights Q and R (corresponding to Q_x and Q_u) in the textbook):

i. Use the "default" choice for Q and R : the identity (select the appropriate dimension for each).

ii. Bryson and Ho (1975) have suggested the following method: Start by choosing Q and R as diagonal matrices whose elements are the inverses of the squares of the maxima of the corresponding variables. (Bryson and Ho suggest that one can then modify the elements by hand to obtain a compromise among response time, damping, and control effort.)

   Use the Bryson and Ho method for an initial guess, assuming that the largest value of x_1 is 5, the largest value of x_2 is 0.75, and the largest control signal is 10. (You do not need to modify the system model to impose these constraints.)

iii. When control actions are very costly, we use a value of R which is relatively higher than the entries of Q . Use a Q with equal penalty to each of the states, and an R which is 10 times the value of the entries of Q.

(b) For each of the choices of Q and R, plot the closed loop response of the system starting from an initial condition $x(0) = (0, 0)$ and show that the system response converges to the desired output value.

(c) Assume now that $d = 0.3$. Show that the system response with your LQR controller from part (b-i) no longer converges to the desired output value.

(d) Construct a controller using integral action that recovers the ability to drive the system state to that point.

(e) Suppose that there is uncertainty in the input matrix, so that the B matrix becomes

$$B = \begin{pmatrix} 0 \\ \gamma \end{pmatrix}, \qquad 0.5 \le \gamma \le 2.$$

Show that the uncertain system response with your original LQR controller no longer converges to the desired equilibrium (even with $d = 0$), but that the controller with integral action still causes the system to converge to the desired output value. (OK to just show this for $\gamma = 0.5$ and 2.)

A Jupyter notebook template that will lead you through this problem is available here.

The following problem is a conceptual problem that you should be able to do without using the textbook, your notes, or the Internet. It is representative of the type of problem you can expect on the (closed-book) final exam (so you might want to first attempt solving it without using any notes).

Problem 3. Consider the system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x,$$

with the control law

$$u = -k_1 x_1 - k_2 x_2 + k_f r.$$

(a) Compute the rank of the reachability matrix for the system and show that it is not reachable.

(b) Compute the characteristic polynomial of the closed loop system and show that the eigenvalues of the system cannot be assigned to arbitrary values.

(Make sure to show your work to receive full credit.)