CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 110/ChE 105

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Spring 2024 | | Due: 24 Apr 2024

Problem 1. In this problem, we explore the influence of the sensor matrix, C, on the output of an LTI system. Consider the following two systems:

$$\Sigma_1\colon \quad \begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} -1.1 & -0.1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \\ y &= \begin{pmatrix} 1.01 & 0.11 \end{pmatrix} x, \end{aligned} \qquad \Sigma_2\colon \quad \begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} -1.1 & -0.1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \\ y &= \begin{pmatrix} 1.1 & 1.01 \end{pmatrix} x. \end{aligned}$$

(a) Find the eigenvalues and eigenvectors for the dynamics matrix A for the systems, and identify the eigenvector along which the open loop dynamics evolve the fastest.

Since A has distinct eigenvalues, the eigenvectors form a basis of $\mathbb{R}^2$, and the state x can be expressed in the eigenspace coordinates $x = Tz$, where the columns of T are the eigenvectors of the dynamics matrix. Conceptually, each element z_i tracks the contribution of the eigenvector v_i to the state x. The output dynamics $y = Cx$ can thus be expressed in the eigenspace coordinates: $y = \tilde{C}z = CTz$.

(b) For each system, compute $\tilde{C}_i = C_i T$. Which mode (eigenvalue) is the largest contributor to the output response of each system?

(c) Plot the step response of each system and compute the 10% to 90% rise time for each system.

(d) Plot the frequency response for each system and identify the input frequency for each system at which the amplitude of the output is a a factor of 2 smaller than the input.

Problem 2. Consider a linear system with dynamics matrix

$$A = \begin{pmatrix} -\zeta\omega_0 & \omega_\mathrm{d} \\ -\omega_\mathrm{d} & -\zeta\omega_0 \end{pmatrix},$$

where $\omega_\mathrm{d} = \omega_0\sqrt{1-\zeta^2}$ and $\zeta < 1$.

(a) Compute the eigenvalues of the dynamics matrix and give conditions on the parameters ω_0 and ζ under which the origin is an asymptotically stable equilibrium point for the linear system $\dot{x} = Ax$.

(b) Show that A can be written in the form $A = S + D$ where S is a skew-symmetric matrix ($S^T = -S$) and D is a diagonal matrix.

(c) It can be shown that if two matrices M_1 and M_2 commute ($M_1M_2 = M_2M_1$) then $\exp(M_1 + M_2) = \exp(M_1)\exp(M_2)$. Use this fact to show that

$$\exp \begin{pmatrix} -\zeta\omega_0 & \omega_\mathrm{d} \\ -\omega_\mathrm{d} & -\zeta\omega_0 \end{pmatrix} t = e^{-\zeta\omega_0 t} \begin{pmatrix} \cos\omega_\mathrm{d}t & \sin\omega_\mathrm{d}t \\ -\sin\omega_\mathrm{d}t & \cos\omega_\mathrm{d}t \end{pmatrix}.$$

This computation is the basis of how the matrix exponential is computed in the case when the eigenvalues have an imaginary component. It can be used to compute the response for a underdamped spring mass system, where ζ is the damping ratio of the system.

Problem 3. Consider the following linear systems with $x \in \mathbb{R}^2$:

$$\Sigma_1\colon \quad \dot{x} = \begin{pmatrix} 0 & 1 \\ -a & -b \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

$$\Sigma_2\colon \quad \dot{x} = \begin{pmatrix} 0 & 1 \\ -a & b \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

$$\Sigma_3\colon \quad \dot{x} = \begin{pmatrix} 0 & 0 \\ -a & -b \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

Assume that $a > 0$ and $b > 0$.

(a) Determine whether each system is stable, asymptotically stable, or unstable. Justify your answer.

(b) Consider the step responses shown below. For each step response, indicate which system it corresponds to, and justify your answer.