CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 110/ChE 105

Problem Set #2

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Problem 1. Keynes' simple model for an economy is given by

$$Y[k] = C[k] + I[k] + G[k],$$

where Y, C, I, and G are gross national product (GNP), consumption, investment, and government expenditure for year k. Consumption and investment are modeled by difference equations of the form

$$C[k+1] = aY[k], \qquad I[k+1] = b(C[k+1] - C[k]),$$

where a and b are parameters. The first equation implies that consumption increases with GNP but that the effect is delayed. The second equation implies that investment is proportional to the rate of change of consumption.

Show that the equilibrium value of the GNP is given by

$$Y_{\mathrm{e}} = \frac{1}{1-a} G_{\mathrm{e}},$$

where the parameter $1/(1-a)$ is the Keynes multiplier (the gain from G to Y). With $a = 0.75$ an increase of government expenditure will result in a fourfold increase of GNP. Also show that the model can be written as the following discrete-time state model:

$$\begin{pmatrix} C[k+1] \\ I[k+1] \end{pmatrix} = \begin{pmatrix} a & a \\ ab-b & ab \end{pmatrix} \begin{pmatrix} C[k] \\ I[k] \end{pmatrix} + \begin{pmatrix} a \\ ab \end{pmatrix} G[k],$$
$$Y[k] = C[k] + I[k] + G[k].$$

Problem 2. Consider a nonlinear system with dynamics

$$m\ddot{q} = -k(q - aq^3) - c\dot{q},$$

where m, k, a, and c are parameters with positive values. Note that this is similar to the spring mass system we have studied in Section 3.2 (and in class), except for the nonlinearity.

(a) Rewrite the system in state space form, i.e. as a system of two first-order ODEs.

(b) Find the equilibrium points of this system.

(c) Linearize the system and check the stability at each equilibrium point for the following parameters: $m = 1000$, $k = 250$, $a = 0.01$, and $c = 100$.

(d) Plot a phase portrait for the system, identifying the equilibrium points, separatrices (if any), limit cycles (if any), and streamlines. Include a sufficient number of initial conditions to clearly illustrate the nonlinear dynamics and appropriate regions of attraction.

(e) Using the phase portrait, find two initial conditions $x_1(0)$ and $x_2(0)$ that satisfy $\|x_1(0) - x_2(0)\| < 0.1$ but whose time responses diverge. Illustrate your results by plotting the time responses for the two initial conditions you found and also plotting those two trajectories on your phase plot (in a unique color, so they are clearly identifiable).

Problem 3. Let $A \in \mathbb{R}^{n\times n}$ be a square matrix with real eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding eigenvectors $v_1, \dots, v_n$. Assume that the eigenvalues are distinct ($\lambda_i \neq \lambda_j$ for $i \neq j$) and real-valued.

Note: The first two parts make use of properties from basic linear algebra, but you should show why the statements are correct.

(a) Show that $v_i \neq v_j$ for $i \neq j$.

(b) Show that the eigenvectors form a basis for $\mathbb{R}^n$ so that any vector x can be written as $x = \sum \alpha_i v_i$ for $\alpha_i \in \mathbb{R}$.

(c) Let $T = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}$ and show that $T^{-1}AT$ is a diagonal matrix of the form (5.10) in FBS2e.

(d) Show that if we rewrite the differential equation in a new set of coordinates $z = T^{-1}x$ that the dynamics are given by $\dot{z}_i = \lambda_i z_i$, $i = 1, \dots, n$.

This form of the dynamics is called *modal form*, with each v_i corresponding to the "mode shape" associated with eigenvalue λ_i. As we shall see when we study linear systems next week (or you may remember from your class on differential equations), the solution to a system in modal form can be written as a superposition of the solutions for each mode i.

If the eigenvalues are complex, it can be shown that A can be transformed to be in the form:

$$A = \begin{pmatrix} \Lambda_1 & & 0 \\ & \ddots & \\ 0 & & \Lambda_k \end{pmatrix}, \quad \text{where} \quad \Lambda_i = \lambda \in \mathbb{R} \quad \text{or} \quad \Lambda_i = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$$

This form of the dynamics of a linear system is often referred to as *block diagonal form.*

Problem 4. Consider the cruise control system described in Section 4.1 of FBS2e. Generate a phase portrait for the closed loop system on flat ground ($\theta = 0$), in fourth gear, using a PI controller (with $k_\text{p} = 0.5$ and $k_\text{i} = 0.1$), $m = 1600$ kg, and desired speed 20 m/s. Your system model should include the effects of saturating the input between 0 and 1.

(Hint: Keep in mind that when modeling feedback control, additional states can arise that do not appear in the original dynamics. You should include the Python code used to generate your phase portrait.)

Note: there is a bug in v0.10.0 of python-control with the way that interconnected systems are handled, so you need to either load the development version of python-control, as shown in Friday's lecture, or write the closed loop dynamics as a single nonlinear I/O system, without using the `interconnect` *function. (This will be fixed in python-control 0.10.1.)*

The following problem is a conceptual problem that you should be able to do without using the textbook, your notes, or the Internet. It is representative of the type of problem you can expect on the (closed-book) final exam (so you might want to first attempt solving it without using any notes).

Problem 5. Consider a nonlinear model of population dynamics given by

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right), \qquad x \geq 0,$$

where k is the *carrying capacity* of the environment. This model is called the *logistic growth model*. Determine all equilibrium points, their stability (including any dependence on the values of r and k), and the linearizations around the equilibrium points.