L9-3: PID Control of Servomechanism

CDS 110/ChE 105, Winter 2024
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In this problem we will use a variety of methods to design proportional (P), proportional-integral (PI), and proportional-integral-derivative (PID) controllers for a cart pendulum system.

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from math import pi

try:
  import control as ct
  print("python-control", ct.__version__)
except ImportError:
  # Get the development version, which fixes a bug that affects the code bel
  !pip install git+https://github.com/python-control/python-control.git
  import control as ct
```

```
python-control 0.10.0
```

System dynamics

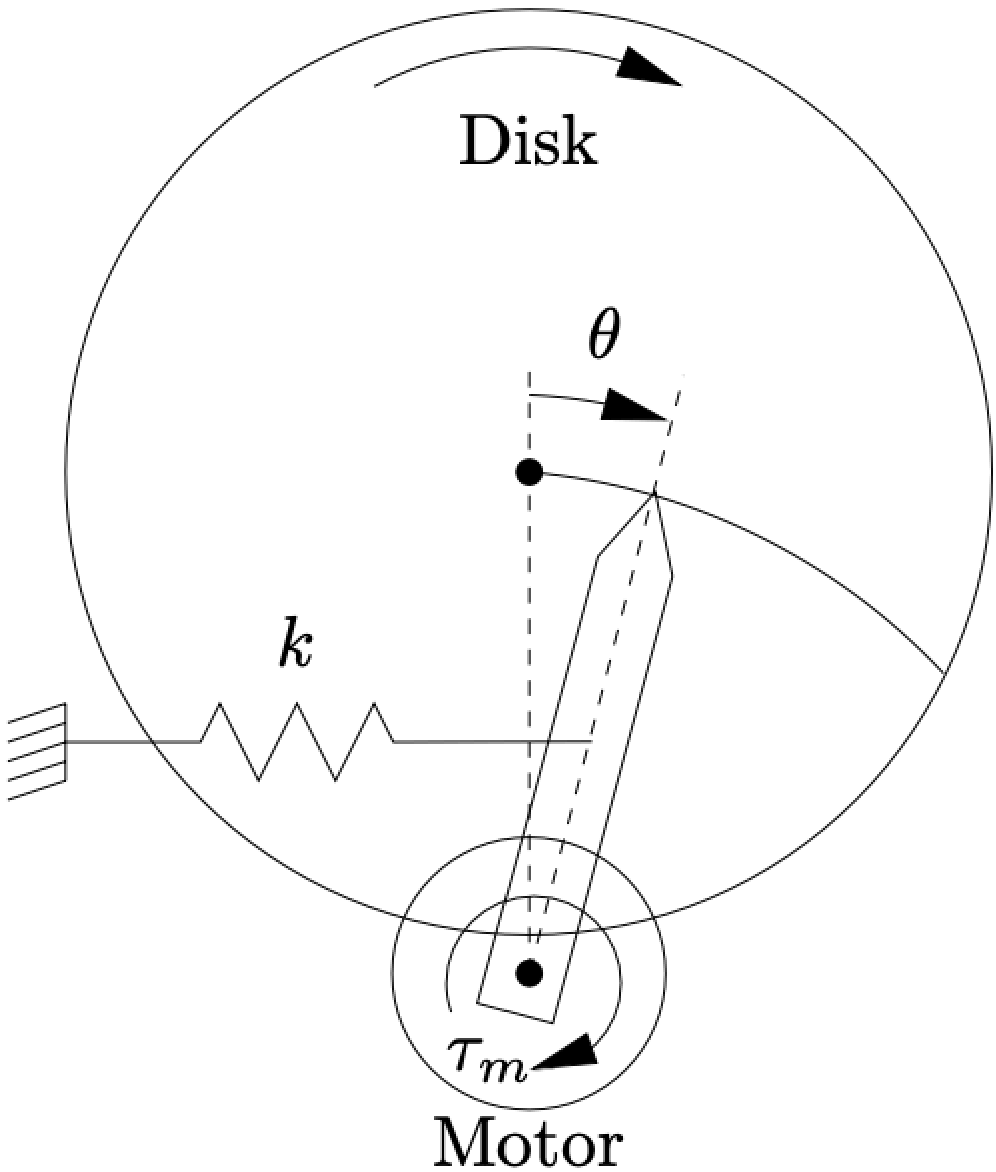
Consider a simple mechanism for positioning a mechanical arm whose equations of motion are given by

$$J\ddot{\theta} = -b\dot{\theta} - kr\sin\theta + \tau_\mathrm{m},$$

which can be written in state space form as

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \theta\end{bmatrix} = \begin{bmatrix}\dot{\theta} \\ -kr\sin\theta/J - b\dot{\theta}/J\end{bmatrix} + \begin{bmatrix}0 \\ 1/J\end{bmatrix}\tau_\mathrm{m}.$$

The system consists of a spring loaded arm that is driven by a motor, as shown below.



The motor applies a torque that twists the arm against a linear spring and moves the end of the arm across a rotating platter. The input to the system is the motor torque τ_{m}. The force exerted by the spring is a nonlinear function of the head position due to the way it is attached.

The system parameters are given by

$$k = 1, \quad J = 100, \quad b = 10, \quad r = 1, \quad l = 2, \quad \epsilon = 0.01.$$

and we assume that time is measured in msec and distance in cm. (The constants here are made up and don't necessarily reflect a real disk drive, though the units and time constants are motivated by computer disk drives.)

In [2]:
```
# Parameter values
servomech_params = {
    'J': 100,             # Moment of inertial of the motor
    'b': 10,              # Angular damping of the arm
    'k': 1,               # Spring constant
```

```
    'r': 1,                # Location of spring contact on arm
    'l': 2,                # Distance to the read head
    'eps': 0.01,           # Magnitude of velocity-dependent perturbation
}

# State derivative
def servomech_update(t, x, u, params):
    # Extract the configuration and velocity variables from the state vector
    theta = x[0]                 # Angular position of the disk drive arm
    thetadot = x[1]              # Angular velocity of the disk drive arm
    tau = u[0]                   # Torque applied at the base of the arm

    # Get the parameter values
    J, b, k, r = map(params.get, ['J', 'b', 'k', 'r'])

    # Compute the angular acceleration
    dthetadot = 1/J * (
        -b * thetadot - k * r * np.sin(theta) + tau)

    # Return the state update law
    return np.array([thetadot, dthetadot])

# System output (full state)
def servomech_output(t, x, u, params):
    l = params['l']
    return l * x[0]

# System dynamics
servomech = ct.nlsys(
    servomech_update, servomech_output, name='servomech',
    params=servomech_params,
    states=['theta_', 'thdot_'],
    outputs=['y'], inputs=['tau'])
```

In addition to the system dynamics, we assume there are actuator dynamics that limit the performance of the system. We take these as first order dynamics with saturation:

$$\tau = \mathrm{sat}\left(\frac{\alpha}{s+\alpha} u\right)$$

In [3]:
```
actuator_params = {
    'umax': 5,             # Saturation limits
    'alpha': 10,           # Actuator time constant
}

def actuator_update(t, x, u, params):
  # Get parameter values
  alpha = params['alpha']
  umax = params['umax']

  # Clip the input
  u_clip = np.clip(u, -umax, umax)

  # Actuator dynamics
```

```
    return -alpha * x + alpha * u_clip

actuator = ct.nlsys(
    actuator_update, None, params=actuator_params,
    inputs='u', outputs='tau', states=1, name='actuator')

system = ct.series(actuator, servomech)
system.name = 'system'  # missing feature
print(system)
```

```
<InterconnectedSystem>: system
Inputs (1): ['u[0]']
Outputs (1): ['y[0]']
States (3): ['actuator_x[0]', 'servomech_theta_', 'servomech_thdot_']

Update: <function InterconnectedSystem.__init__.<locals>.updfcn at 0x140d111
c0>
Output: <function InterconnectedSystem.__init__.<locals>.outfcn at 0x140d112
60>
```

Linearization

To study the open loop dynamicsof the system, we compute the linearization of the dynamics about the equilibrium point corresponding to $\theta_\mathrm{e} = 15^\circ$.

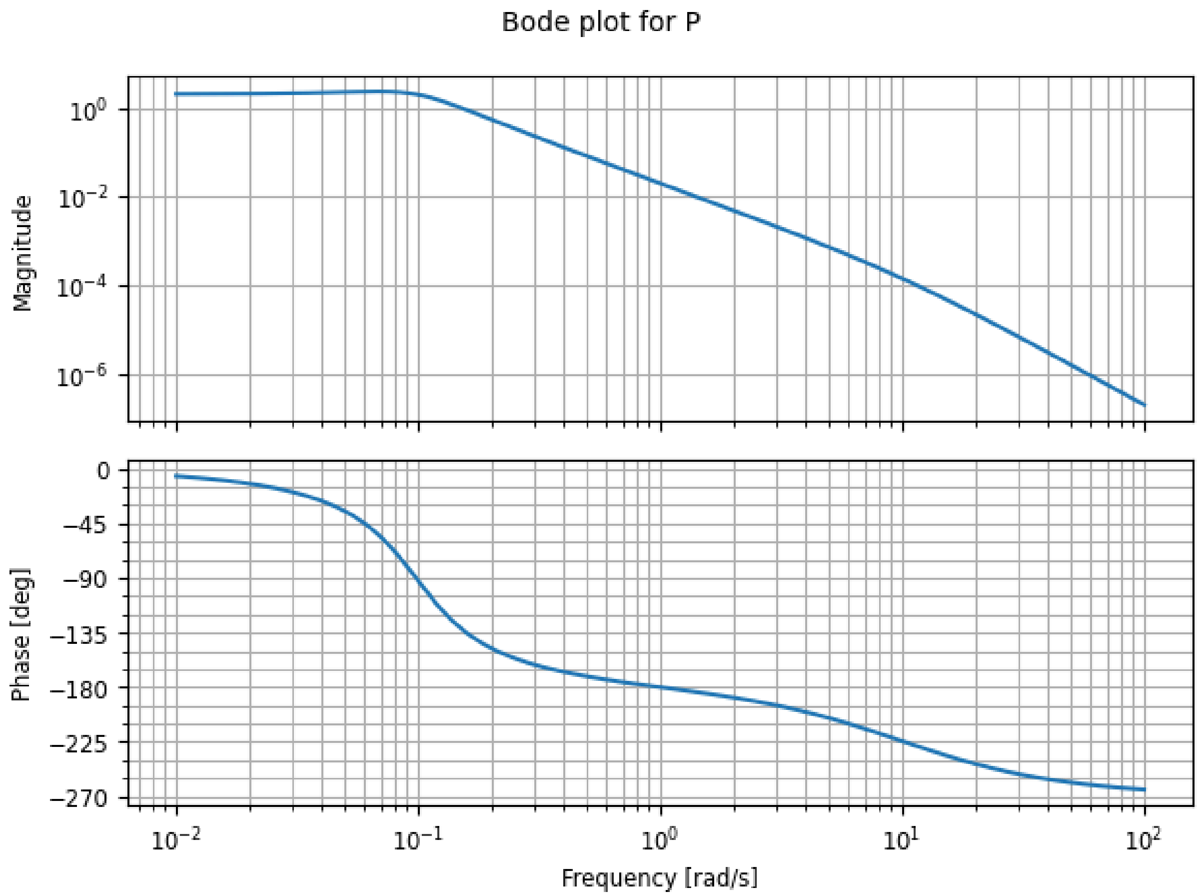
In [4]:

```
# Convert the equilibrium angle to radians
theta_e = (15 / 180) * np.pi

# Compute the input required to hold this position
u_e = servomech.params['k'] * servomech.params['r'] * np.sin(theta_e)
print("Equilibrium torque = %g" % u_e)

# Linearize the system dynamics about the equilibrium point
P = ct.tf(
    system.linearize([0, theta_e, 0], u_e, copy_names=True)[0, 0])
P.name = 'P'  # bug
print(P, end="\n\n")

ct.bode_plot(P)
```

```
Equilibrium torque = 0.258819
<TransferFunction>: P
Inputs (1): ['u[0]']
Outputs (1): ['y[0]']


              0.2
---------------------------------
s^3 + 10.1 s^2 + 1.01 s + 0.09659
```

Out[4]:

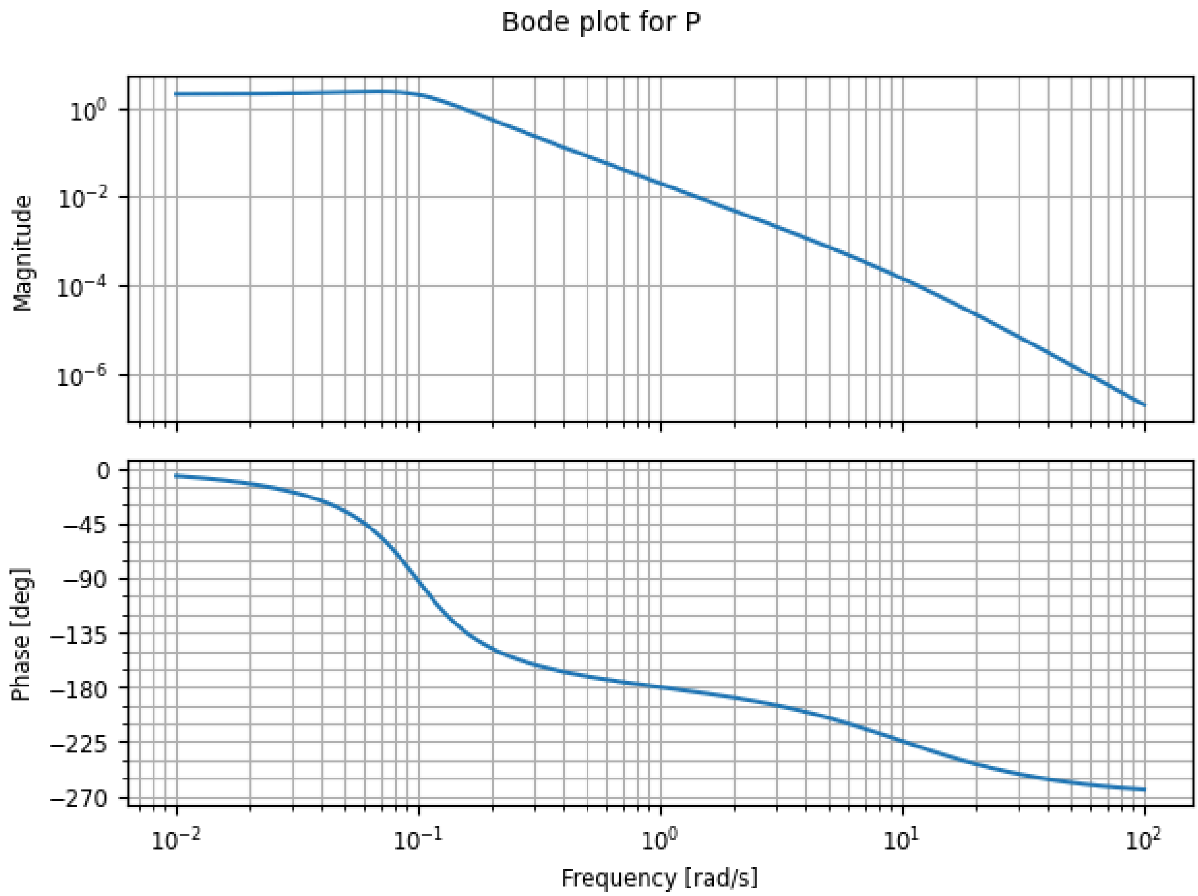
```
array([[list([<matplotlib.lines.Line2D object at 0x140d2aab0>])],
       [list([<matplotlib.lines.Line2D object at 0x140d9fc20>])]],
      dtype=object)
```



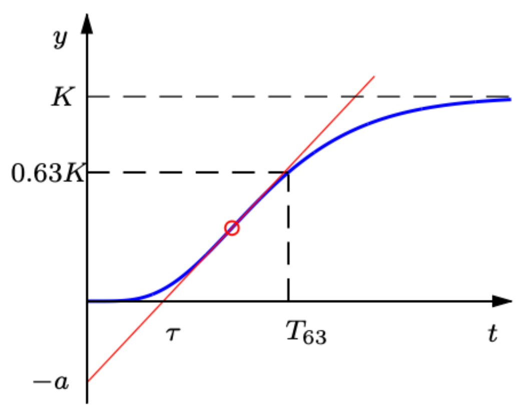
In [5]:
```
# Colab: create a simplified form to get rid of warning messages
if len(P.num[0][0]) > 1:
  P = ct.tf(P.num[0][0][2], P.den, name='P')  # Fix up conditioning
print(P)
ct.bode_plot(P)
```

```
<TransferFunction>: P
Inputs (1): ['u[0]']
Outputs (1): ['y[0]']


               0.2
---------------------------------
s^3 + 10.1 s^2 + 1.01 s + 0.09659
```

Out[5]:
```
array([[list([<matplotlib.lines.Line2D object at 0x14157b260>])],
       [list([<matplotlib.lines.Line2D object at 0x14157b590>])]],
      dtype=object)
```



Ziegler-Nichols tuning



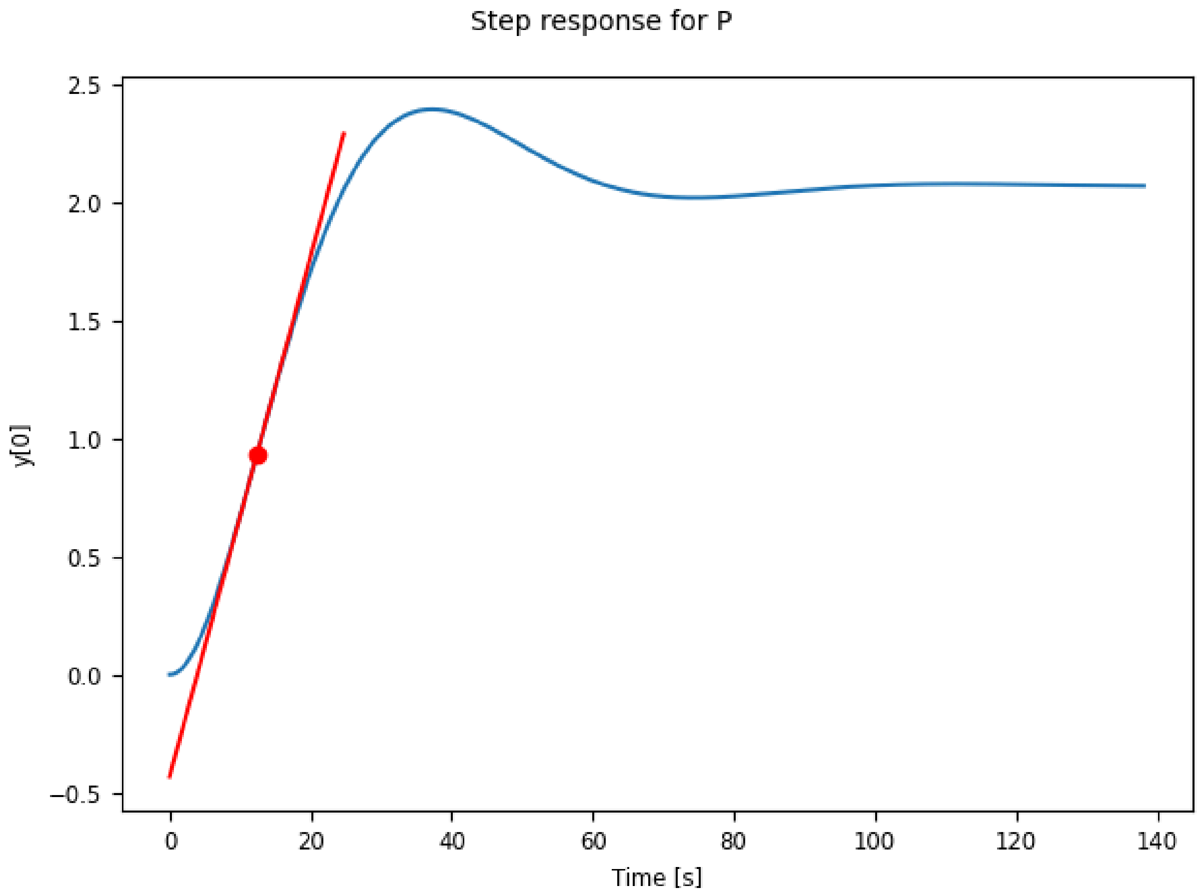
(a) Step response method

Type	k_p	T_i	T_d
P	$1/a$		
PI	$0.9/a$	$\tau/0.3$	
PID	$1.2/a$	$\tau/0.5$	0.5τ

In [6]:
```
# Plot the step response
resp = ct.step_response(P)
resp.plot()

# Find the point of the steepest slope
slope = np.diff(resp.outputs) / np.diff(resp.time)
mxi = np.argmax(slope)
mx_time = resp.time[mxi]
```

```
mx_out= resp.outputs[mxi]
plt.plot(resp.time[mxi], resp.outputs[mxi], 'ro')

# Draw a line going through the point of max slope
mx_slope = slope[mxi]
timepts = np.linspace(0, mx_time*2)
plt.plot(timepts, mx_out + mx_slope * (timepts - mx_time), 'r-')

# Solve for the Ziegler-Nichols parameters
a = -(mx_out - mx_slope * mx_time)  # Find the value of the line at t = 0
tau = a / mx_slope                  # Solve a + mx_slope * tau = 0
print(f"{a=}, {tau=}")
```

```
a=0.43326263676972254, tau=3.9297636758958916
```



In [7]:
```
s = ct.tf('s')

# Proportional controller
kp = 1/a
ctrl_zn_P = kp

# PI controller
kp = 0.9/a
Ti = tau/0.3; ki = kp/Ti
ctrl_zn_PI = kp + ki / s

# PID controller
kp = 1.2/a
Ti = tau/0.5; ki = kp/Ti
```

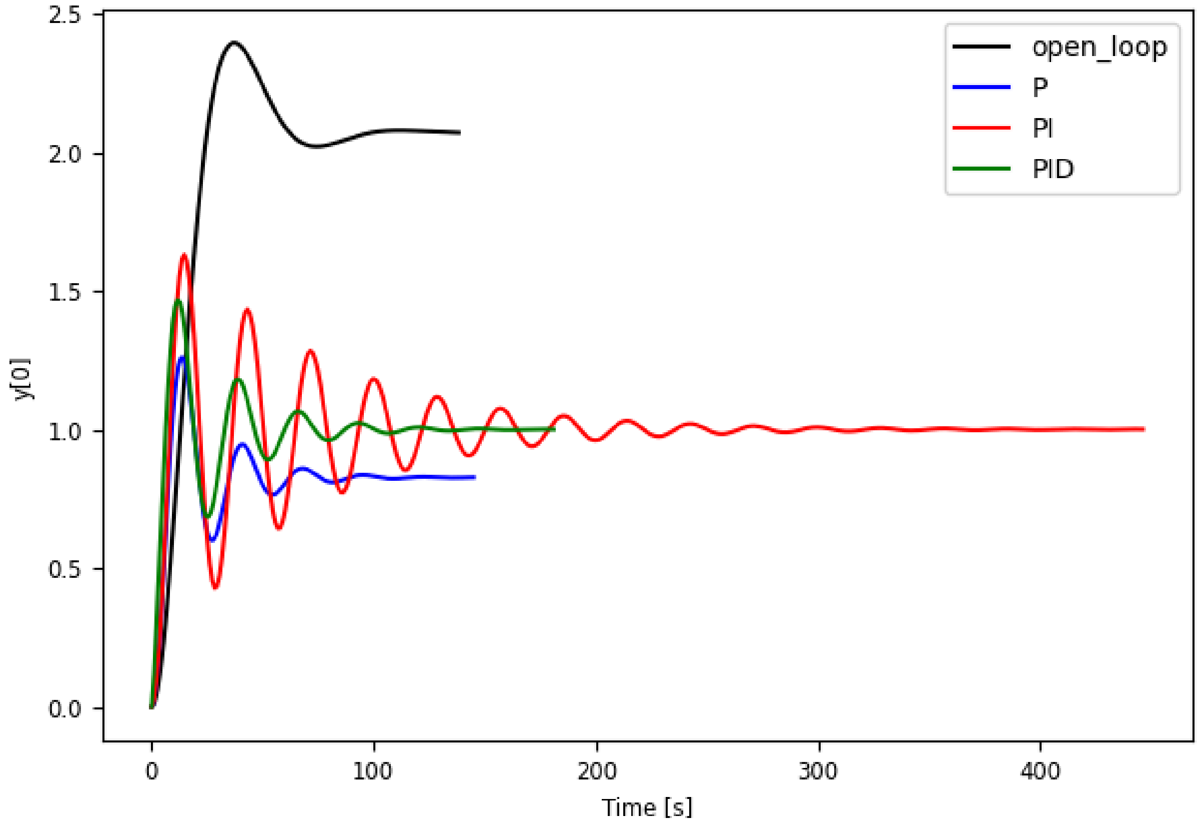
```python
Td = 0.5 * tau; kd = kp * Td
ctrl_zn_PID = kp + ki / s + kd * s

print(ctrl_zn_PID)
```

```
<TransferFunction>: sys[19]
Inputs (1): ['u[0]']
Outputs (1): ['y[0]']


5.442 s^2 + 2.77 s + 0.3524
---------------------------
             s
```

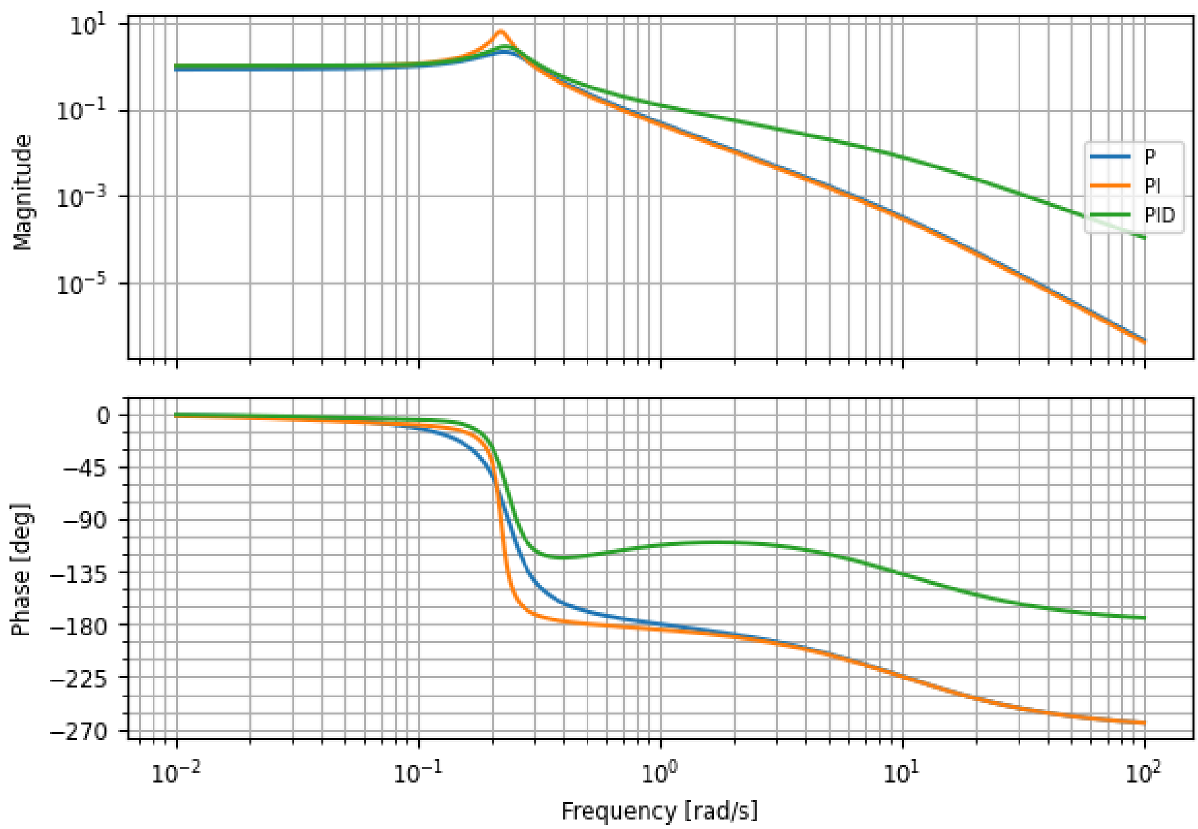
In [8]:

```python
# Compute the closed loop systems and plots the step and
# frequency responses.

clsys_zn_P = ct.feedback(P * ctrl_zn_P)
clsys_zn_P.name = 'P'

clsys_zn_PI = ct.feedback(P * ctrl_zn_PI)
clsys_zn_PI.name = 'PI'

clsys_zn_PID = ct.feedback(P * ctrl_zn_PID)
clsys_zn_PID.name = 'PID'

# Plot the step responses
resp.sysname = 'open_loop'
resp.plot(color='k')

stepresp_zn_P = ct.step_response(clsys_zn_P)
stepresp_zn_P.plot(color='b')

stepresp_zn_PI = ct.step_response(clsys_zn_PI)
stepresp_zn_PI.plot(color='r')

stepresp_zn_PID = ct.step_response(clsys_zn_PID)
stepresp_zn_PID.plot(color='g')
plt.legend()

plt.figure()
ct.bode_plot([clsys_zn_P, clsys_zn_PI, clsys_zn_PID]);
```

Step response for P, PI, PID

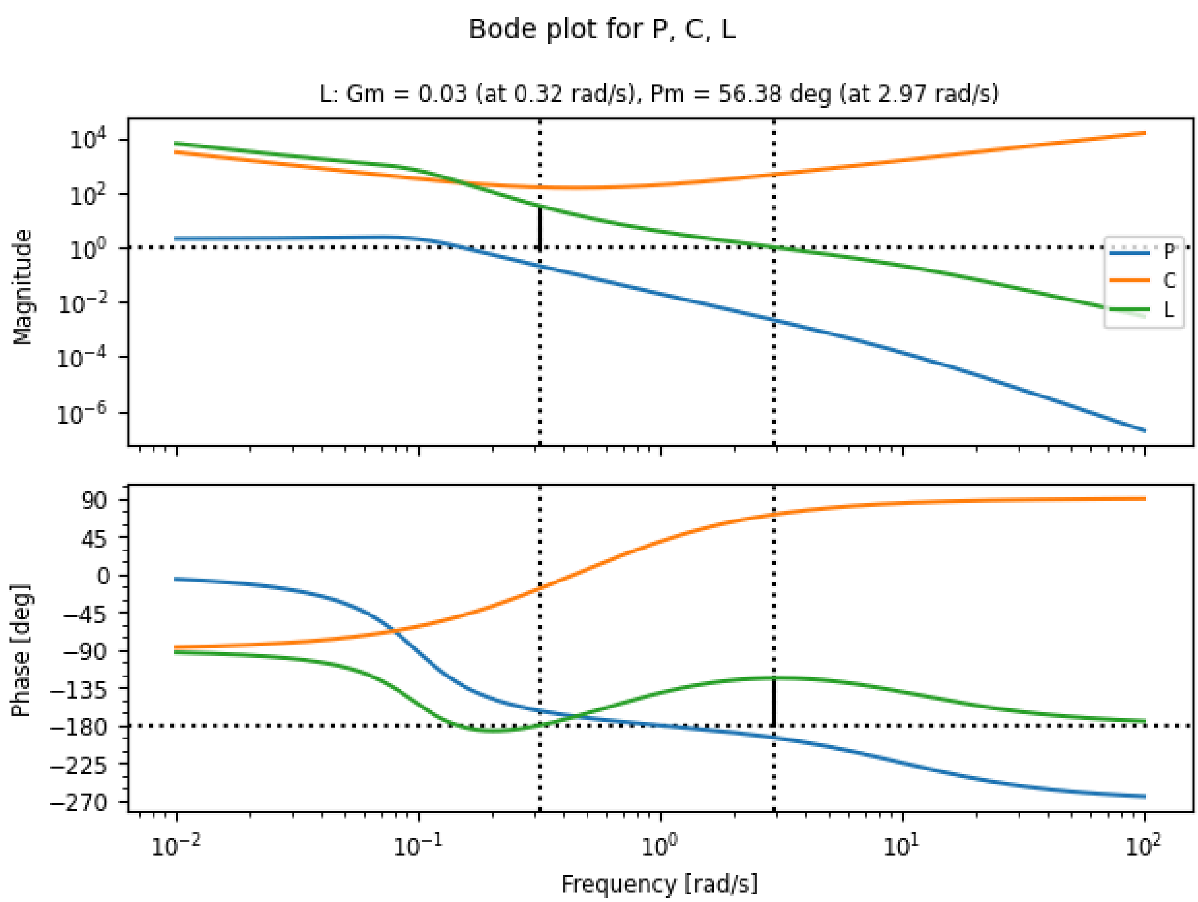


Bode plot for P, PI, PID



Loop shaping

In [9]:
```
# Design parameters
Td = 1                       # Set to gain crossover frequency
Ti = Td * 10                 # Set to low frequency region
kp = 500                     # Tune to get desired bandwith

# Updated gains
kp = 150
Ti = Td * 5; kp = 150

# Compute controller parmeters
ki = kp/Ti
kd = kp * Td

# Controller transfer function
ctrl_shape = kp + ki / s + kd * s
ctrl_shape.name = 'C'

# Frequency response (open loop) - use this to help tune your design
ltf_shape = P * ctrl_shape
ltf_shape.name = 'L'

ct.frequency_response([P, ctrl_shape]).plot()
ct.frequency_response(ltf_shape).plot(margins=True);
```

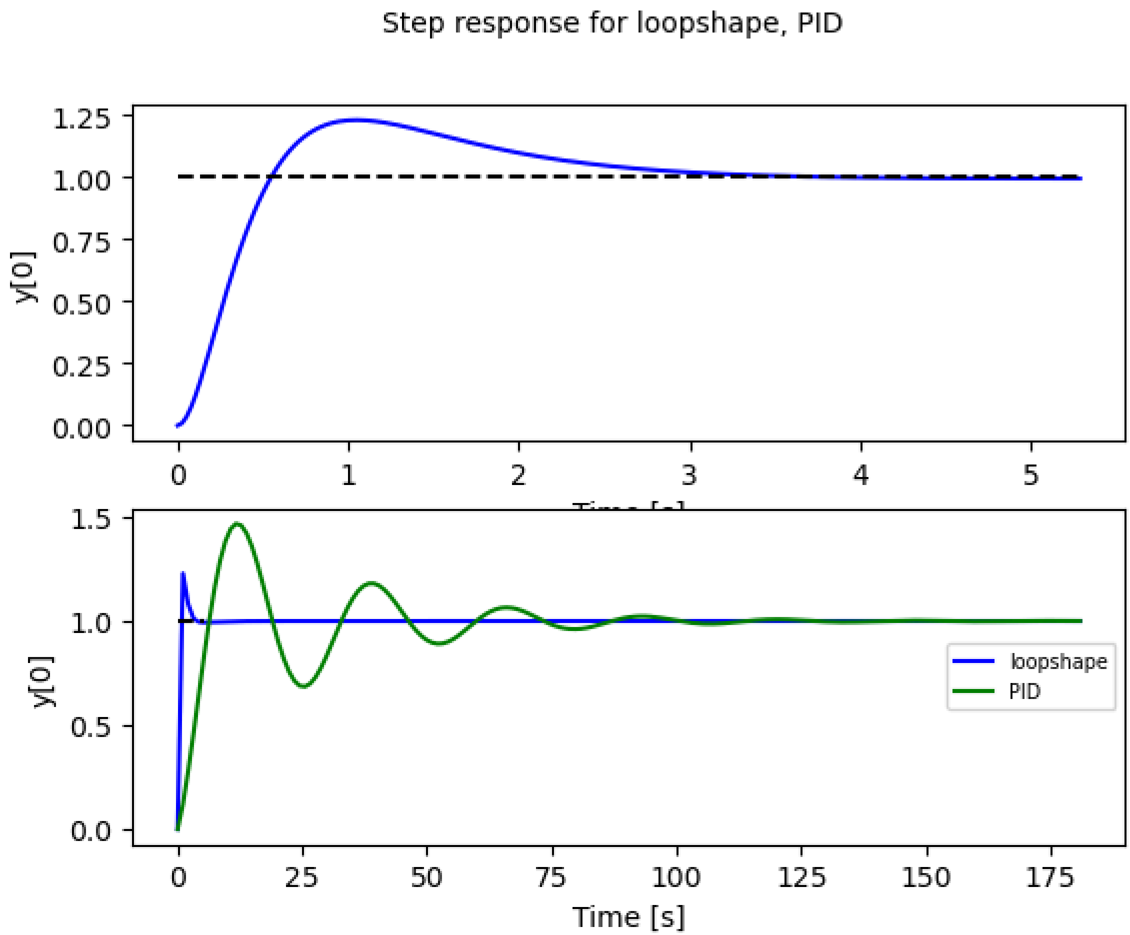


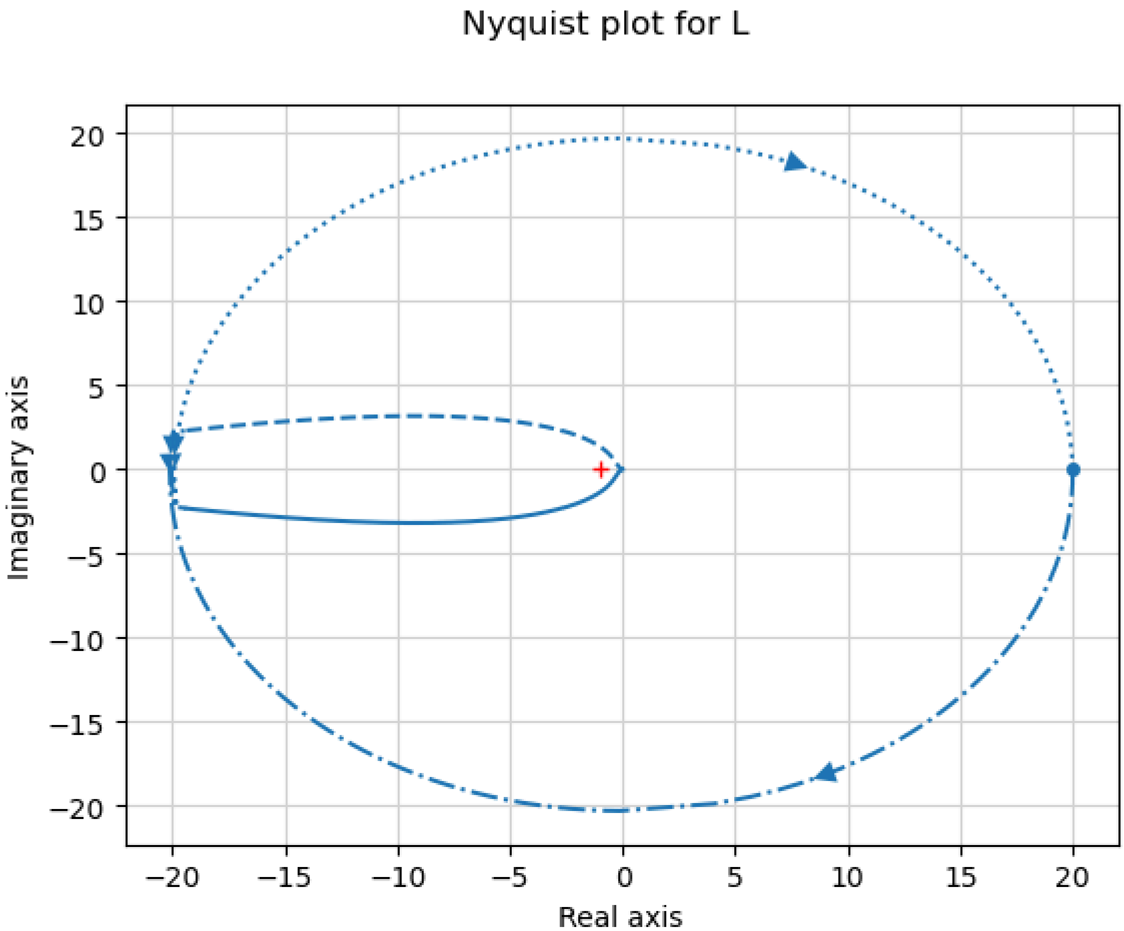
In [10]:

```python
# Compute the closed loop systemsand plot the step response
# and Nyquist plot (to make sure margins look OK)

# Create the closed loop systems
clsys_shape = ct.feedback(P * ctrl_shape)
clsys_shape.name = 'loopshape'

# Step response
plt.subplot(2, 1, 1)
stepresp_shape = ct.step_response(clsys_shape)
stepresp_shape.plot(color='b')
plt.plot([0, stepresp_shape.time[-1]], [1, 1], 'k--')

# Compare to the ZN controller
ax = plt.subplot(2, 1, 2)
ct.step_response(clsys_shape, stepresp_zn_PID.time).plot(
    color='b', ax=np.array([[ax]]))
stepresp_zn_PID.plot(color='g', ax=np.array([[ax]]))
ax.plot([0, stepresp_shape.time[-1]], [1, 1], 'k--')

# Nyquist plot
plt.figure()
ct.nyquist([ltf_shape])
```

Out[10]:

```
array([list([<matplotlib.lines.Line2D object at 0x142ef8050>, <matplotlib.l
ines.Line2D object at 0x142ef9c70>, <matplotlib.lines.Line2D object at 0x14
2fb8dd0>, <matplotlib.lines.Line2D object at 0x142fb95e0>])],
      dtype=object)
```

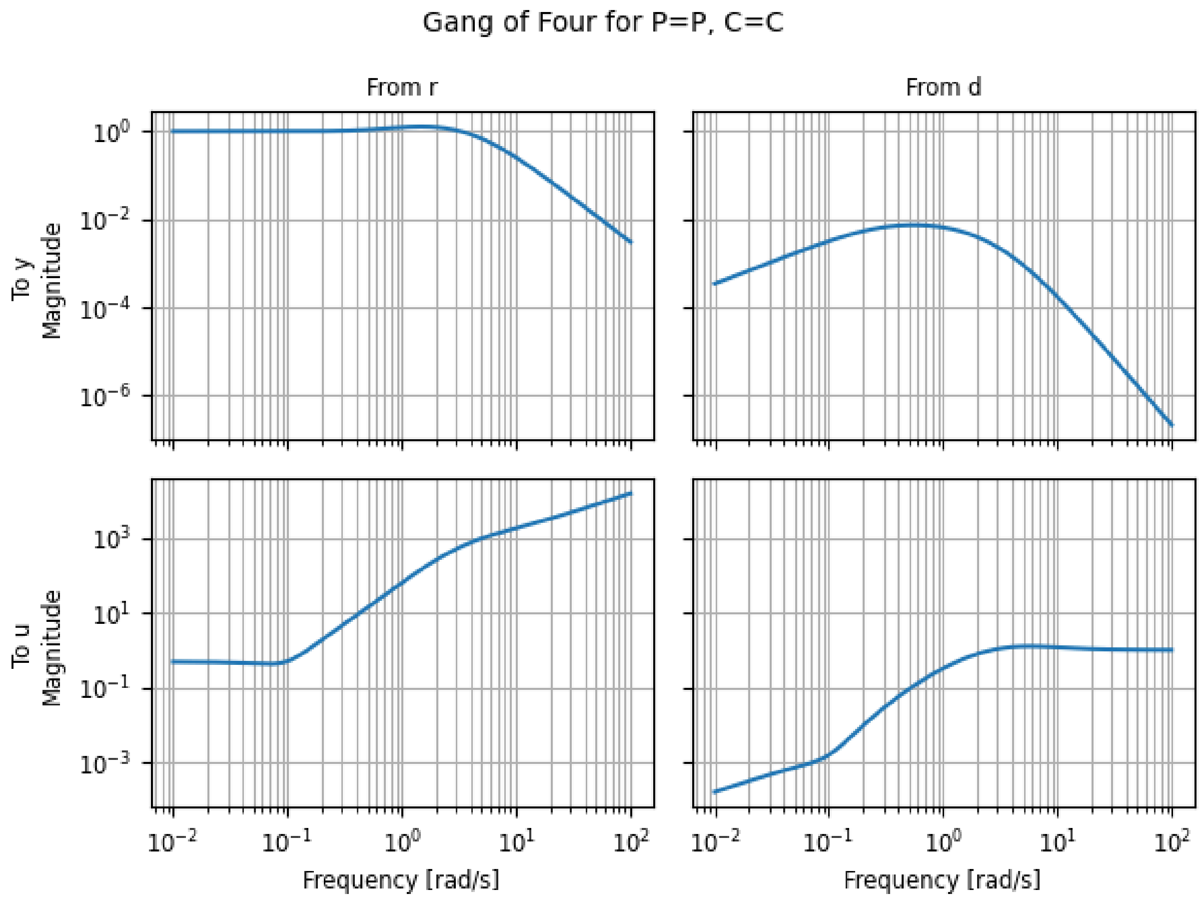

Step response for loopshape, PID
y[0]
1.25
1.00
0.75
0.50
0.25
0.00
0
1
2
3
4
5
y[0]
1.5
1.0
0.5
0.0
0
25
50
75
100
125
150
175
Time [s]
loopshape
PID



Gang of Four

In [11]:
```
ct.gangof4(P, ctrl_shape)
```

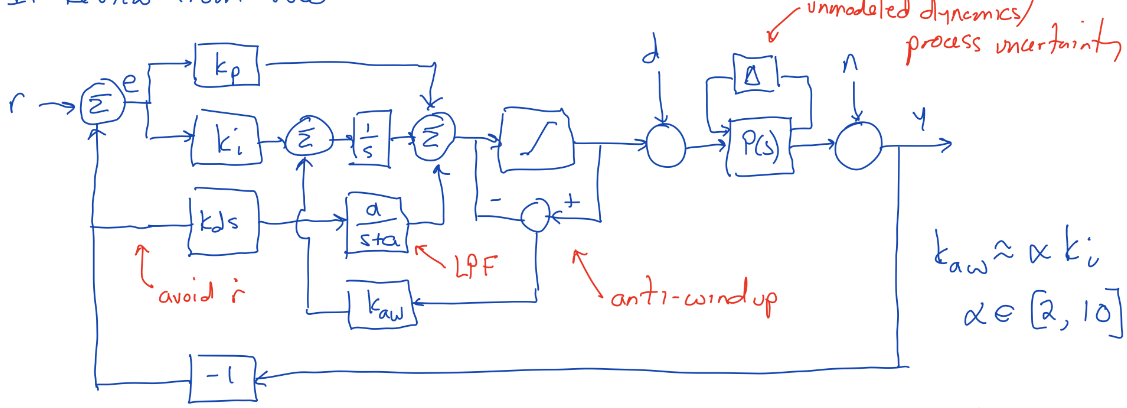
Out[11]:
```
array([[list([<matplotlib.lines.Line2D object at 0x142e665d0>]),
        list([<matplotlib.lines.Line2D object at 0x142e31910>])],
       [list([<matplotlib.lines.Line2D object at 0x142e322d0>]),
        list([<matplotlib.lines.Line2D object at 0x142f49670>])]],
      dtype=object)
```



Anti-windup

We now implement the full PID controller with anti-windup and derivative filtering:

I. Review from Wed



Low pass filter

The low pass filtered derivative has transfer function

$$G(s) = \frac{a\, s}{s + a}.$$

This can be implemented using the differential equation

$$\dot{\xi} = -a\xi + ay, \qquad \eta = -a\xi + ay.$$

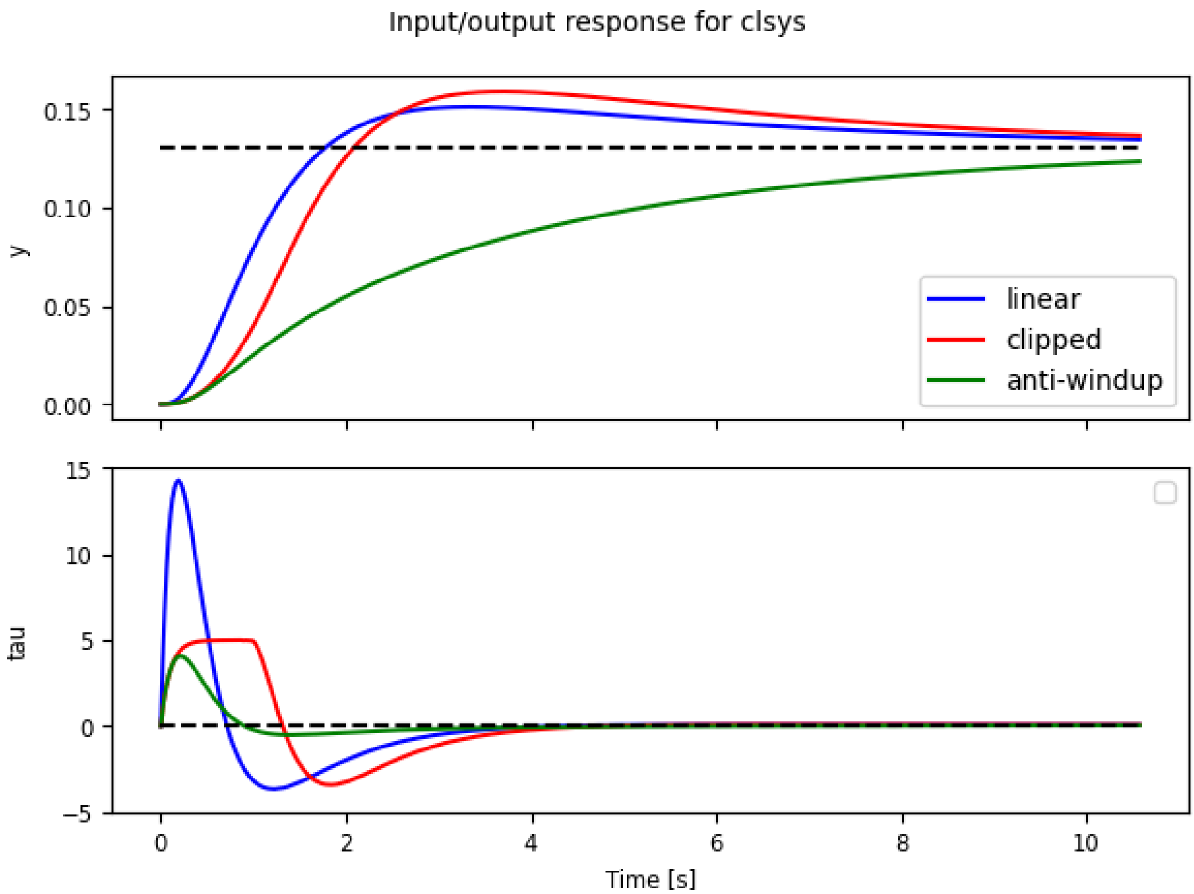
In [12]:
```
ctrl_params = {'kaw': 5 * ki, 'a': 10/Td}

def ctrl_update(t, x, u, params):
  # Get the parameter values
  kaw = params['kaw']
  a = params['a']
  umax_ctrl = params.get('umax_ctrl', actuator.params['umax'])

  # Extract the signals into more familiar variable names
  r, y = u[0], u[1]
  z = x[0]         # integral error
  xi = x[1]        # filtered derivative

  # Compute the controller components
  u_prop = kp * (r - y)
  u_int = z
  ydt_f = -a * xi + a * (-y)
  u_der = kd * ydt_f

  # Compute the commanded and saturated outputs
  u_cmd = u_prop + u_int + u_der
  u_sat = np.clip(u_cmd, -umax_ctrl, umax_ctrl)

  dz = ki * (r - y) + kaw * (u_sat - u_cmd)
  dxi = -a * xi + a * (-y)
  return np.array([dz, dxi])

def ctrl_output(t, x, u, params):
  # Get the parameter values
  kaw = params['kaw']
  a = params['a']
  umax_ctrl = params.get('umax_ctrl', params['umax'])

  # Extract the signals into more familiar variable names
  r, y = u[0], u[1]
  z = x[0]         # integral error
  xi = x[1]        # filtered derivative

  # Compute the controller components
  u_prop = kp * (r - y)
  u_int = z
  ydt_f = -a * xi + a * (-y)
  u_der = kd * ydt_f

  # Compute the commanded and saturated outputs
  u_cmd = u_prop + u_int + u_der
  u_sat = np.clip(u_cmd, -umax_ctrl, umax_ctrl)

  return u_cmd

ctrl = ct.nlsys(
```

```
    ctrl_update, ctrl_output, name='ctrl', params=ctrl_params,
    inputs=['r', 'y'], outputs=['u'], states=2)

clsys = ct.interconnect(
    [servomech, actuator, ctrl], name='clsys',
    inputs=['r'], outputs=['y', 'tau'])
print(clsys)
```

```
<InterconnectedSystem>: clsys
Inputs (1): ['r']
Outputs (2): ['y', 'tau']
States (5): ['servomech_theta_', 'servomech_thdot_', 'actuator_x[0]', 'ctrl_
x[0]', 'ctrl_x[1]']

Update: <function InterconnectedSystem.__init__.<locals>.updfcn at 0x14296e7
a0>
Output: <function InterconnectedSystem.__init__.<locals>.outfcn at 0x142e600
40>
```

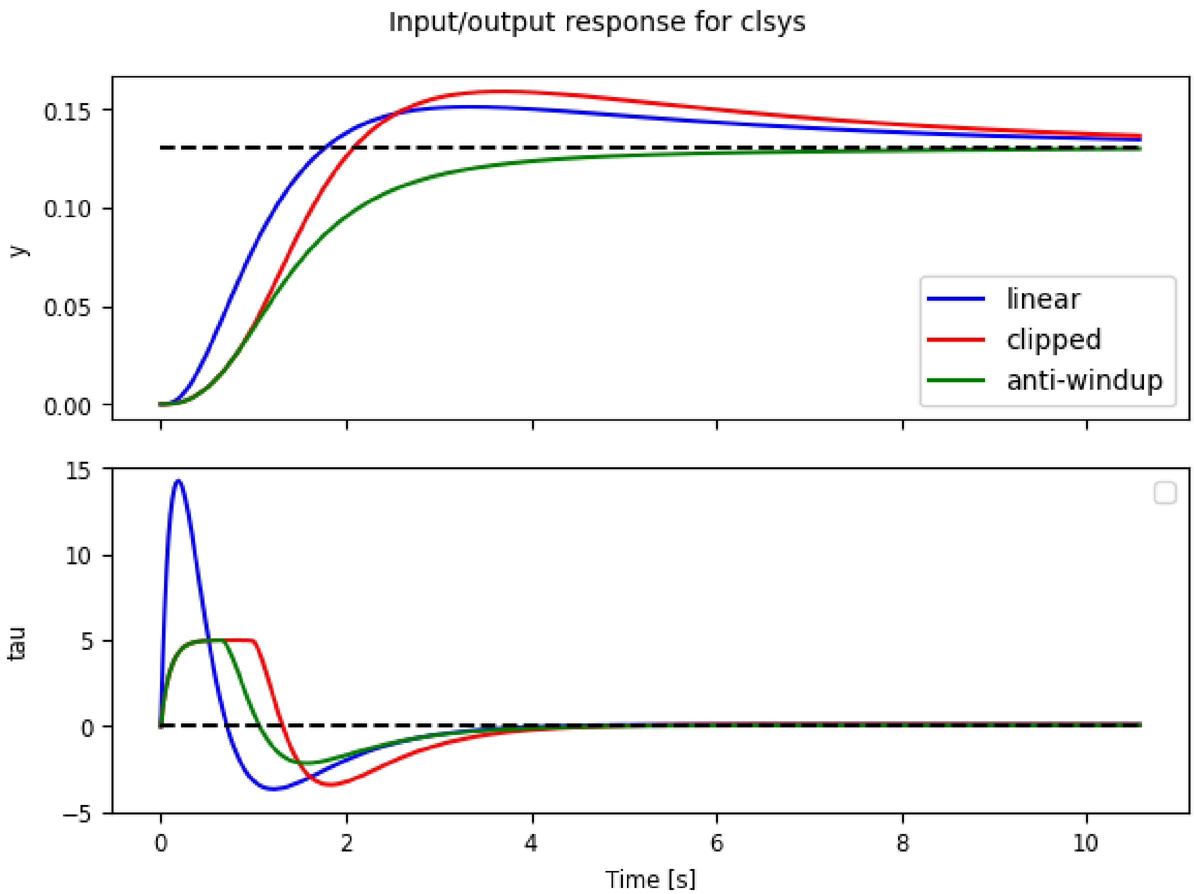
In [13]:

```
# Plot the step responses for the following cases:
#
# 'linear': the original linear step response (no actuation limits)
# 'clipped': PID controller with input limits, but not anti-windup
# 'anti-windup': PID controller with anti-windup compensation

# Use more time points to get smoother response curves
timepts = np.linspace(0, 2*stepresp_shape.time[-1], 500)

# Compute the response for the individual cases
stepsize = theta_e / 2
resp_ln = ct.input_output_response(
    clsys, timepts, stepsize, params={'umax': np.inf, 'kaw': 0, 'a': 1e3})
resp_cl = ct.input_output_response(
    clsys, timepts, stepsize, params={'umax': 5, 'kaw': 0, 'a': 100})
resp_aw = ct.input_output_response(
    clsys, timepts, stepsize, params={'umax': 5, 'kaw': 2*ki, 'a': 100})

# Plot the time responses in a single plot
out = ct.time_response_plot(resp_ln, color='b', plot_inputs=False)
ct.time_response_plot(resp_cl, color='r', plot_inputs=False)
ct.time_response_plot(resp_aw, color='g', plot_inputs=False)

# Annotations
axs = ct.get_plot_axes(out)
axs[0, 0].legend(['linear', 'clipped', 'anti-windup'])
axs[0, 0].plot([0, timepts[-1]], [stepsize, stepsize], 'k--')
axs[1, 0].plot([0, timepts[-1]], [0, 0], 'k--')
axs[1, 0].set_ylim([-5, 15])
axs[1, 0].legend([]);
```



The response of the anti-windup compensator is very sluggish, indicating that we may be setting $k_{\rm aw}$ too high.

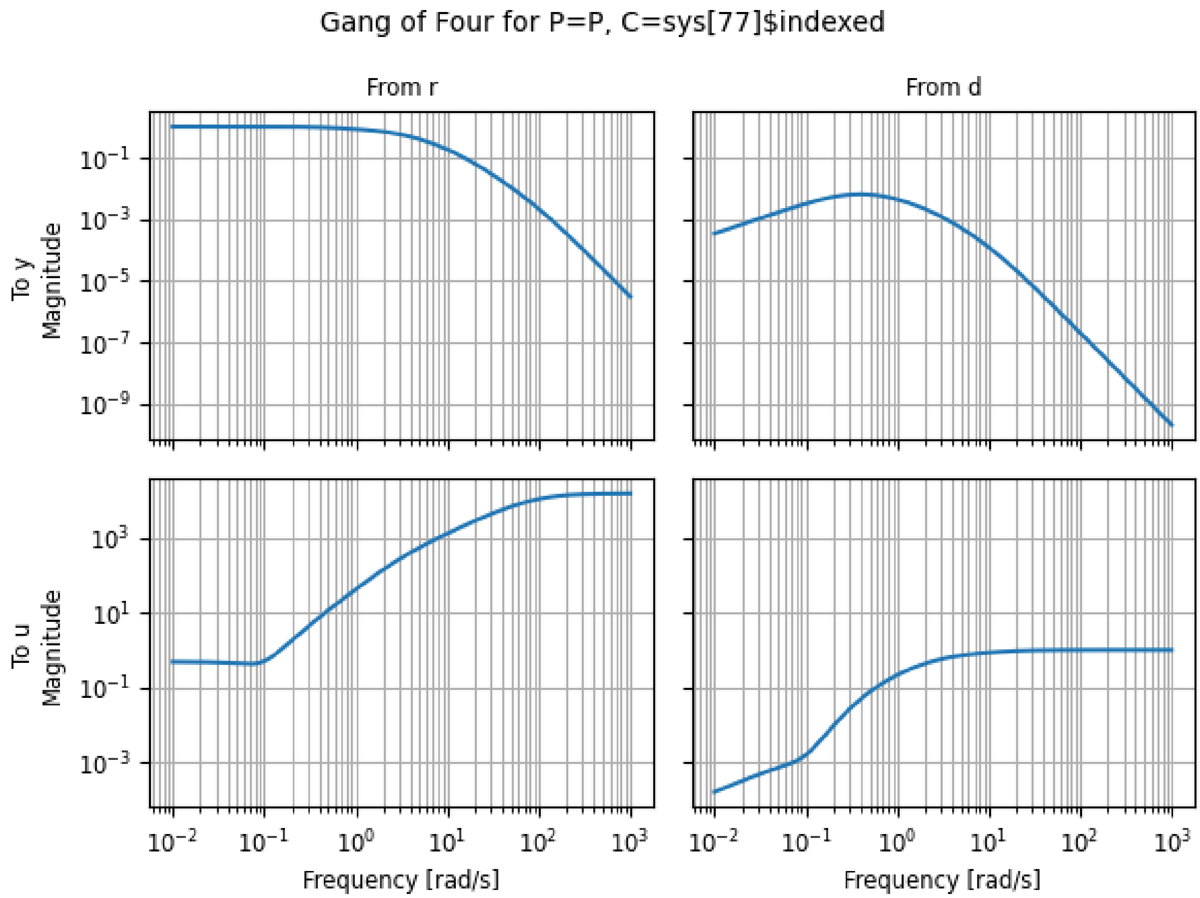
In [14]:
```
resp_aw = ct.input_output_response(
#    clsys, timepts, stepsize, params={'umax': 5, 'kaw': 1, 'a': 100})
    clsys, timepts, stepsize, params={'umax': 5, 'kaw': 0.05 * ki, 'a': 100}

# Plot the time responses in a single plot
out = ct.time_response_plot(resp_ln, color='b', plot_inputs=False)
ct.time_response_plot(resp_cl, color='r', plot_inputs=False)
ct.time_response_plot(resp_aw, color='g', plot_inputs=False)

# Annotations
axs = ct.get_plot_axes(out)
axs[0, 0].legend(['linear', 'clipped', 'anti-windup'])
axs[0, 0].plot([0, timepts[-1]], [stepsize, stepsize], 'k--')
axs[1, 0].plot([0, timepts[-1]], [0, 0], 'k--')
axs[1, 0].set_ylim([-5, 15])
axs[1, 0].legend([]);
```



This gives a much better response, though the value of k_{aw} falls well outside the range of [2, 10]. One reason for this is that k_{aw} acts on the inputs, τ, which are roughly 100 larger than the size of the outputs, y, as seen in the above plots.

We can now see if this affects the Gang of Four in the expected way:

In [15]:
```
C = ctrl.linearize([0, 0], 0, params=resp_aw.params)[0, 1]
ct.gangof4(P, C);
```



Note that in the transfer function from r to u (which is the same as the transfer function from e to u, the high frequency gain is now bounded. (We could make it go back down by using a second order filter.)

```
In [ ]:
```