Goals:

- Show how to use “loop shaping” using PID to achieve a performance specification
- Discuss the use of integral feedback and anti-windup compensation

Reading:

- Åström and Murray, *Feedback Systems*, Ch 11
“Loop Shaping”: Design Loop Transfer Function

Translate specs to “loop shape”

\[ L(s) = P(s)C(s) \]

- Design \( C(s) \) to obey constraints

Typical loop constraints

- High gain at low frequency
  - Good tracking, disturbance rejection at low freqs
- Low gain at high frequency
  - Avoid amplifying noise
- Sufficiently high bandwidth
  - Good rise/settling time
- Shallow slope at crossover
  - Sufficient phase margin for robustness, low overshoot

Key constraint: slope of gain curve determines phase curve

- Can’t independently adjust
- Eg: slope at crossover sets PM

\[ H_{er} = \frac{1}{1 + L} \quad H_{\eta m} = \frac{-L}{1 + L} \]
Gain Crossover Specifications

High phase at gain crossover can lead to poor performance

- System can be stable but still have bad response at certain frequencies
- Typically occurs if system has low phase margin $\Rightarrow$ get resonant peak in closed loop (Mr) + poor step response
- Solution: specify minimum phase margin. Typically 45° or more

\[
\begin{align*}
H_{yr} &= \frac{L}{1 + L} \\
GM &= \text{Gain Margin} \\
PM &= \text{Phase Margin}
\end{align*}
\]
Overview of Loop Shaping

Performance specification
- Steady state error
- Tracking error
- Bandwidth (possibly from rise/settling time)
- Robustness margins/overshoot

Approach: “shape” loop transfer function using $C(s)$
- $P(s) +$ specifications given
- $L(s) = P(s) C(s)$
  - Use $C(s)$ to choose desired shape for $L(s)$
- Important: can’t set gain and phase independently
  - Slope at crossover determines phase
Overview: PID control

Intuition

• Proportional term: provides inputs that correct for “current” errors
• Integral term: insures steady state error goes to zero
• Derivative term: provides “anticipation” of upcoming changes

A bit of history on “three term control”

• First appeared in 1922 paper by Minorsky: “Directional stability of automatically steered bodies” under the name “three term control”
• Also realized that “small deviations” (linearization) could be used to understand the (nonlinear) system dynamics under control

Utility of PID

• PID control is most common feedback structure in engineering systems
• For many systems, only need PI or PD (special case)
• Many tools for tuning PID loops and designing gains

\[ u = k_p e + k_i \int e \, dt + k_d \dot{e} \]
Proportional Feedback

Simplest controller choice: $u = k_p e$

- Effect: lifts gain with no change in phase
- Good for plants with low phase up to desired bandwidth
- Bode: shift gain up by factor of $k_p$
- Step response: better steady state error, but with decreasing stability

![Diagram of proportional feedback system](image)

![Graphs showing response to proportional feedback](graphs)
Proportional + Integral Compensation

Use to eliminate steady state error

- Effect: lifts gain at low frequency
- Gives zero steady state error
- Bode: infinite SS gain + phase lag
- Step response: zero steady state error, with smaller settling time, but more overshoot

\[ r \rightarrow + e \rightarrow k_p + \frac{k_i}{s} \rightarrow u \rightarrow P(s) \rightarrow y \]

\[ k_p > 0, \quad k_i > 0 \]

\[ \omega_z = K_I / K_p \]

\( L(s) \quad C(s) \)

\( P(s) \)

\( \text{Output } y \)

\( \text{Input } u \)

\( k_i \)
**Proportional + Integral + Derivative (PID)**

The transfer function of a PID controller is given by:

\[
C(s) = k_p + \frac{k_i}{s} + k_d s
\]

or equivalently,

\[
C(s) = k_p (1 + \frac{1}{T_i s} + T_d s)
\]

or

\[
\approx T_d k_p \frac{(s + 1/T_i)(s + 1/T_d)}{s}
\]

The Bode diagrams illustrate the phase and magnitude characteristics of the PID controller.

**Bode Diagrams**

- **Phase (deg); Magnitude (dB)**
- **Frequency (rad/sec)**

The diagrams show the frequency responses with critical frequencies:

\[
\omega_1 = \frac{1}{T_i}, \quad \omega_2 = \frac{1}{T_d}
\]
Implementing Derivative Action

Problems with derivatives
- High frequency noise amplified by derivative term
- Step inputs in reference can cause large inputs
- Show up in Gang of Four...

Solution: modified PID control
- Use high frequency rolloff in derivative term
  - first order filter will give finite gain at high frequency
  - use higher order filter if needed
- Don’t feed reference signal through derivative block
  - Useful when reference has unwanted high frequency content
  - Better solution: reference shaping via two DOF design (F(s) block)
- Many other variations (see AM08 + refs)
Example: Cruise Control using PID - Specification

**Performance Specification**

- \( \leq 1\% \) steady state error
  - Zero frequency gain > 100
- \( \leq 10\% \) tracking error up to 10 rad/sec
  - Gain > 10 from 0-10 rad/sec
- \( \geq 45^\circ \) phase margin
  - Gives good relative stability
  - Provides robustness to uncertainty

**Observations**

- Purely proportional gain won’t work: to get gain above desired level will not leave adequate phase margin
- Need to increase the phase from \(~0.5\) to 2 rad/sec and increase gain as well

\[
P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s + a}
\]
Example: Cruise Control using PID - Design

Approach
- Use integral gain to make steady state error small (zero, in fact)
- Use derivative action to increase phase lead in the cross over region
- Use proportional gain to give desired bandwidth

Controller
- \( Ti = 1/0.1; \quad Td = 1/1; \quad k = 2000 \)

\[
C(s) = \frac{2000s^2 + 1.1s + 0.1}{s} = 2200 + \frac{200}{s} + 2000s
\]

Closed loop system
- Very high steady state gain
- Adequate tracking @ 1 rad/sec
- \( \sim 80^\circ \) phase margin
- Verify with Nyquist
Example: Cruise Control using PID - Verification

\[ P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s + a} \]

\[ C(s) = \frac{2000s^2 + 1.1s + 0.1}{s} \]

**Observations**
- *Very fast response (probably too aggressive)*
- Back off on Ti to get something more reasonable
# PID Tuning

## Zeigler-Nichols step response method
- Design PID gains based on step response
- Measure maximum slope + intercept
- Works OK for many plants (but underdamped)
- Good way to get a first cut controller

## Zeigler-Nichols frequency response method
- Increase gain until system goes unstable
- Use critical gain and frequency as parameters

## Variations
- Modified formulas (see text) give better response
- Relay feedback: provides automated way to obtain critical gain, frequency

<table>
<thead>
<tr>
<th>Type</th>
<th>$k_p$</th>
<th>$T_i$</th>
<th>$T_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>1/a</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PI</td>
<td>0.9/a</td>
<td>3τ</td>
<td></td>
</tr>
<tr>
<td>PID</td>
<td>1.2/a</td>
<td>2τ</td>
<td>0.5τ</td>
</tr>
</tbody>
</table>

### (a) Step response method

<table>
<thead>
<tr>
<th>Type</th>
<th>$k_p$</th>
<th>$T_i$</th>
<th>$T_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>0.5$k_c$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PI</td>
<td>0.4$k_c$</td>
<td>0.8$T_c$</td>
<td></td>
</tr>
<tr>
<td>PID</td>
<td>0.6$k_c$</td>
<td>0.5$T_c$</td>
<td>0.125$T_c$</td>
</tr>
</tbody>
</table>

### (b) Frequency response method

\[
k_p = \frac{0.15\tau + 0.35T}{K\tau}\left(\frac{0.9T}{K\tau^2}\right), \quad k_i = \frac{0.46\tau + 0.02T}{K\tau^2}\left(\frac{0.3T}{K\tau^2}\right),
\]

\[
k_p = \frac{0.22k_c - 0.07}{K}\left(\frac{0.4k_c}{T_c}\right), \quad k_i = \frac{0.16k_c}{T_c} + \frac{0.62}{KT_c}\left(\frac{0.5k_c}{T_c}\right).
\]
Example: PID cruise control

Ziegler-Nichols design for cruise controller

- Plot step response, extract $\tau$ and $a$, compute gains

$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s + a}$$

**Bode Diagrams**

- Frequency (rad/sec)
- Phase (deg); Magnitude (dB)

**Step Response**

- Time (sec.)
- Amplitude

$$C(s) = K \left( 1 + \frac{1}{T_i s} + T_D s \right)$$

- Result: sluggish $\Rightarrow$ increase loop gain + more phase margin (shift zero)

$$K = \frac{1.2}{a} \quad T_i = 2 \times \tau \quad T_d = \tau / 2$$
Windup and Anti-Windup Compensation

**Problem**
- Limited magnitude input (saturation)
- Integrator “winds up” => overshoot

**Solution**
- Compare commanded input to actual
- Subtract off difference from integrator
Summary: Frequency Domain Design using PID

Loop Shaping for Stability & Performance
- Steady state error, bandwidth, tracking

\[ H_{ue}(s) = K_p + K_I \frac{1}{s} + K_D s \]

Main ideas
- Performance specs give bounds on loop transfer function
- Use controller to shape response
- Gain/phase relationships constrain design approach
- Standard compensators: proportional, PI, PID