Goals:
- Review canonical control design problem / standard performance measures
- Show how to use “loop shaping” to achieve a performance specification
- Discuss fundamental limits on performance: algebraic + Bode integral
  - More on Wed: maximum modulus principle
- Work through some simple examples of a control design problem

Reading:
Input/Output Control Design Specifications

$F(s) = 1$: Four unique transfer functions define performance (“Gang of Four”)
- Stability is always determined by $1/(1+PC)$ assuming stable process & controller
- Numerator determined by forward path between input and output

More generally: 6 primary transfer functions; simultaneous design of each
- Controller $C(s)$ enters in multiple places ⇒ hard to understand tradeoffs

Design represents a tradeoff between the quantities
- Keep $L=PC$ large for good performance ($H_{er} << 1$)
- Keep $L=PC$ small for good noise rejection ($H_{nn} < 1$)

Keep track all input/output transfer functions
- Keep error small for all reference signals $r$
- Attenuate effect of sensor noise $n$ and disturbances $d$
- Avoid large input cmds $u$

$\begin{bmatrix} \eta \\ y \\ u \end{bmatrix} = \begin{bmatrix} \frac{P}{1+PC} & \frac{PC}{1+PC} & \frac{PCF}{1+PC} \\ \frac{P}{1+PC} & \frac{1}{1+PC} & \frac{PCF}{1+PC} \\ -\frac{PC}{1+PC} & -\frac{C}{1+PC} & \frac{CF}{1+PC} \end{bmatrix} \begin{bmatrix} d \\ n \\ r \end{bmatrix}$
Frequency Domain Specifications

Specifications on the *open loop* transfer function \((L)\)

- Gain crossover frequency, \(\omega_{gc}\), is the lowest frequency at which loop gain = 1
- Gain margin, \(g_m\), is the amount the loop gain can be increased before instability
- Phase margin, \(\phi_m\), is amount of phase lag required to generate instability

Specifications on *closed loop* frequency response (eg \(H_{yr}, H_{yd}\,\text{etc}\))

- Resonant peak, \(M_r\), is the largest value of the frequency response
- Peak frequency, \(\omega_p\), is the frequency where the maximum occurs
- Bandwidth, \(\omega_b\), is the frequency where the gain has decreased to 1/\(\sqrt{2}\)

Basic idea: convert specs on closed loop to specs on open loop

- Bandwidth \(\approx\) value for which \(|L| = 1\)
- Resonant peak set by phase margin
- Keep \(L\) large to set \(H_{yr} \approx 1\)

\[
H_{yr} = \frac{L}{1 + L} \quad H_{er} = \frac{1}{1 + L}
\]
Time Domain Specs $\rightarrow$ Frequency Domain Specs

Time domain specifications

Map to frequency domain for second order system

$$L(s) = \frac{k}{s^2 + bs}$$

$$H_{yr} = \frac{k}{s^2 + bs + k}$$

- Use properties of 2nd order systems (Table 7.1)
**“Loop Shaping”: Design Loop Transfer Function**

Translate specs to “loop shape”

\[ L(s) = P(s)C(s) \]

- Design \( C(s) \) to obey constraints

**Typical loop constraints**

- High gain at low frequency
  - Good tracking, disturbance rejection at low freqs
- Low gain at high frequency
  - Avoid amplifying noise
- Sufficiently high bandwidth
  - Good rise/settling time
- Shallow slope at crossover
  - Sufficient phase margin for robustness, low overshoot

**Key constraint: slope of gain curve determines phase curve**

- Can’t independently adjust
- Eg: slope at crossover sets PM
Example: Lead Compensation for Second Order System

System description

\[ P(s) = \frac{p_1 p_2}{(s + p_1)(s + p_2)} \]

- Poles: \( p_1 = 1, \ p_2 = 5 \)

Control specs

- Track constant reference with error < 1%
- Good tracking up to 100 rad/s (less than 10% error)
- Overshoot less than 10%
  - Gives PM of ~60 deg

Try a “lead” compensator

\[ C(s) = K \frac{s + a}{s + b} \]

- Want gain crossover at approximately 100 rad/sec => center phase gain there
- Set zero frequency gain of controller to give small error => \( |L(0)| > 100 \)
- \( a = 20, \ b = 500, \ K = 10,000 \) (gives \( |C(0)| = |L(0)| = 400 \)
Safety Check: Nyquist + Gang of 4

Nyquist verifies closed loop stability
- Infinite GM; good phase margin

Gang of 4 shows high noise sens’y
- Factor of 10,000 gain at high freq
- Step responses show similar sensitivity

Solution: high freq rolloff (L7-3)
Algebraic Constraints on Performance

Goal: keep $S$ & $T$ small
- $S$ small $\Rightarrow$ low tracking error
- $T$ small $\Rightarrow$ good noise rejection (and robustness [CDS 131/221])

Problem: $S + T = 1$
- Can’t make both $S$ & $T$ small at the same frequency
- Solution: keep $S$ small at low frequency and $T$ small at high frequency
- Loop gain interpretation: keep $L$ large at low frequency, and small at high frequency

\[
H_{er} = \frac{1}{1 + PC} =: S
\]

\[
H_{yn} = \frac{PC}{1 + PC} =: T
\]
Bode’s Integral Formula and the Waterbed Effect

Bode’s integral formula for \( S = \frac{1}{1+PC} = \frac{1}{1+L} \):
- Let \( p_k \) be the unstable poles of \( L(s) \) and assume relative degree of \( L(s) \geq 2 \)
- Theorem: the area under the sensitivity function is a conserved quantity:

\[
\int_{0}^{\infty} \log_e |S(j\omega)| \, d\omega = \int_{0}^{\infty} \log_e \left| \frac{1}{1 + L(j\omega)} \right| \, d\omega = \pi \sum \text{Re} \, p_k
\]

Waterbed effect:
- Making sensitivity smaller over some frequency range requires *increase* in sensitivity someplace else
- Presence of RHP poles makes this effect worse
- Actuator bandwidth further limits what you can do
- Note: area formula is linear in \( \omega \); Bode plots are logarithmic
Example: Magnetic Levitation

System description
- Ball levitated by electromagnet
- Inputs: current thru electromagnet
- Outputs: position of ball (from IR sensor)
- States: \( z, \dot{z} \)
- Dynamics: \( F = ma, F = \text{magnetic force generated by wire coil} \)
- See MATLAB handout for details

Controller circuit
- Active R/C filter network
- Inputs: set point, disturbance, ball position
- States: currents and voltages
- Outputs: electromagnet current
Equations of Motion

Process: actuation, sensing, dynamics

\[ m\ddot{z} = mg - k_m(k_A u)^2/z^2 - c\dot{z} \]
\[ v_{ir} = k_T\dot{z} + v_0 \]

- \( u \) = current to electromagnet
- \( v_{ir} \) = voltage from IR sensor

Linearization:

\[ P(s) = \frac{-k}{s^2 + cs - r^2} \quad k, c, r > 0 \]

- Poles at \( s = \pm r \Rightarrow \) open loop unstable

Note: RHP pole in \( L \Rightarrow \) need one net encirclement (CCW)
Control Design

Need to create encirclement

- Loop shaping is not useful here
- Flip gain to bring Nyquist plot over -1 point
- Insert phase to create CCW encirclement

Can accomplish using a lead compensator

- Produce phase lead at crossover
- Generates loop in Nyquist plot

\[ C(s) = -k \frac{s + a}{s + b} \]
Performance Limits

Nominal design gives low perf
- Not enough gain at low frequency
- Try to adjust overall gain to improve low frequency response
- Works well at moderate gain, but notice waterbed effect

Bode integral limits improvement
\[ \int_0^\infty \log |S(j\omega)|d\omega = \pi r \]
- Must increase sensitivity at some point

![Sensitivity Function](image)

![Step Response](image)
Right Half Plane Zeros

Right half plane zeros produce “non-minimum phase” behavior
- Phase of frequency response has additional phase lag for given magnitude
- Can cause output to move opposite from input for a short period of time

Example: $H_1(s) = \frac{s + a}{s^2 + 2\zeta \omega_n s + \omega_n^2}$ vs $H_2(s) = \frac{s - a}{s^2 + 2\zeta \omega_n s + \omega_n^2}$
Example: Lateral Control of the Ducted Fan

Source of non-minimum phase behavior
- To move left, need to make $\theta > 0$
- To generate positive $\theta$, need $f_1 > 0$
- Positive $f_1$ causes fan to move right initially
- Fan starts to move left after short time (as fan rotates)

Will show on Wed that RHP zeros impose limits on achievable performance

$$H_{xf_1}(s) = \frac{(s^2 - mgl)}{s^2(Js^2 + ds + mgl)}$$

- Poles: 0, 0, $-\sigma \pm j \omega_d$
- Zeros: $\pm \sqrt{mgl}$

Step Response

Fan moves right and then moves to the left
Summary: Limits of Performance

Many limits to performance
- Algebraic: $S + T = 1$
- RHP poles: Bode integral formula
- RHP zeros: Waterbed effect on peak of $S$

Main message: try to avoid RHP poles and zeros whenever possible (eg, re-design)

\[
\int_0^\infty \log_e |S(j\omega)| \, d\omega = \int_0^\infty \log_e \left| \frac{1}{1 + L(j\omega)} \right| \, d\omega = \pi \sum \text{Re} \, p_k
\]
Announcements

Final exam
• Seniors/grad students: 7 Jun (Fri) @ 2 pm
  - Fr, So, Jr can take exam early
• Fr, So, Jr: 12 Jun (Wed) @ 2 pm
• In person, closed book, 1 hour
  - T/F, multiple choice, short answer
  - Similar to final problem on HW

Date changes for final weeks of class
• 27 May: Memorial day => 2 day shift
• HW8: due date 29 May (W) → 31 May (F)
  - W8 office hours: Wed, 3-4 and Thu, 4-5
• W9: 29 May (W), 31 May (F), 3 Jun (M)
• W9 office hours: Wed, 3-4 and Thu, 4-5
  - HW #9 (Fr, So, Jr): due 7 Jun (Fri)
  - HW 9 or questions about class
• Review for final on 5 Jun (Wed)
  - No class on 7 Jun (Fri) [final for Sr/Gr]
• Final OH: 10 Jun (M), 3-4, 11 Jun (Tu), 4-5

YouTube: “Chicken Head Tracking”

2007 DARPA Urban Challenge (“Alice”)