CDS 110/ChE 105: Lecture 8-1
Frequency Domain Design/Limits

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20 May 2024

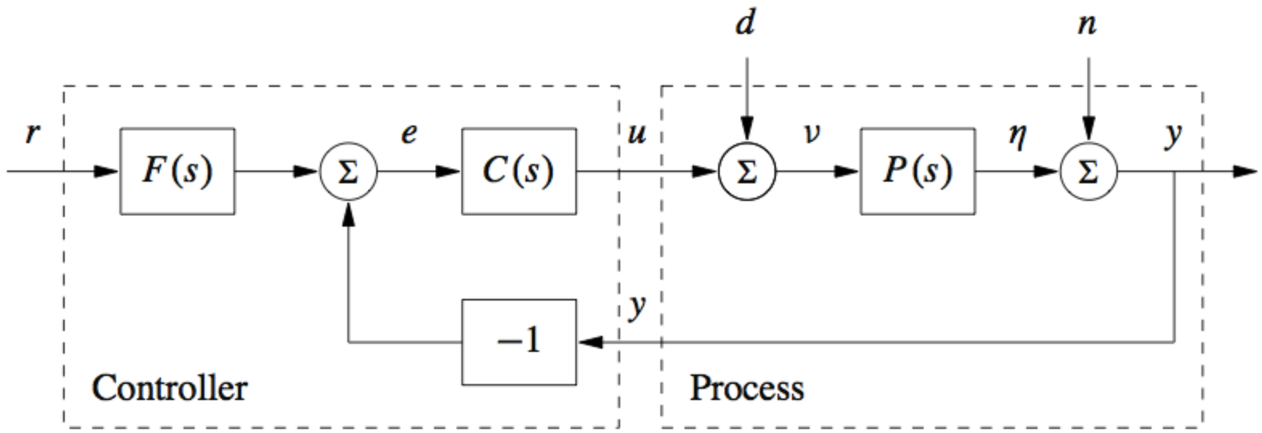
Goals:

- Review canonical control design problem / standard performance measures
- Show how to use “loop shaping” to achieve a performance specification
- Discuss fundamental limits on performance: algebraic + Bode integral
    - More on Wed: maximum modulus principle
- Work through some simple examples of a control design problem

Reading:

- Åström and Murray, Feedback Systems, Section 12.1-12.3, 13.3, 14.2, 14.4

Input/Output Control Design Specifications



$$\begin{bmatrix} \eta \\ y \\ u \end{bmatrix} = \begin{bmatrix} \frac{P}{1+PC} & -\frac{PC}{1+PC} & \frac{PCF}{1+PC} \\ \frac{P}{1+PC} & \frac{1}{1+PC} & \frac{PCF}{1+PC} \\ -\frac{PC}{1+PC} & -\frac{C}{1+PC} & \frac{CF}{1+PC} \end{bmatrix} \begin{bmatrix} d \\ n \\ r \end{bmatrix}$$

Keep track all input/output transfer functions

- Keep error small for all reference signals r
- Attenuate effect of sensor noise n and disturbances d
- Avoid large input cmds u

Design represents a tradeoff between the quantities

- Keep L=PC large for good performance (H_{er} << 1)
- Keep L=PC small for good noise rejection ($H_{\eta n}$ < 1)

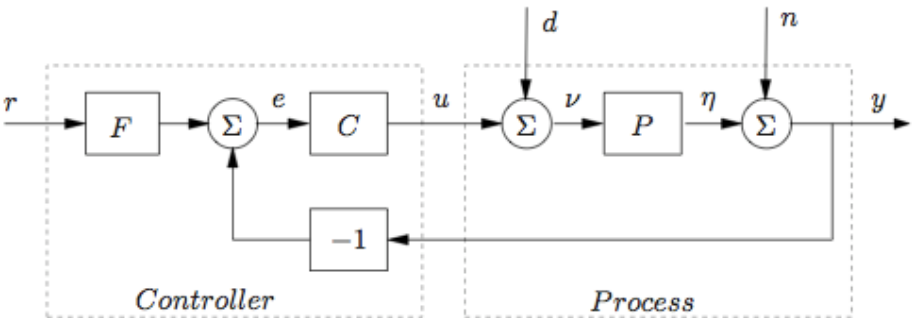
***F*(*s*) = 1: Four unique transfer functions define performance ("Gang of Four")**

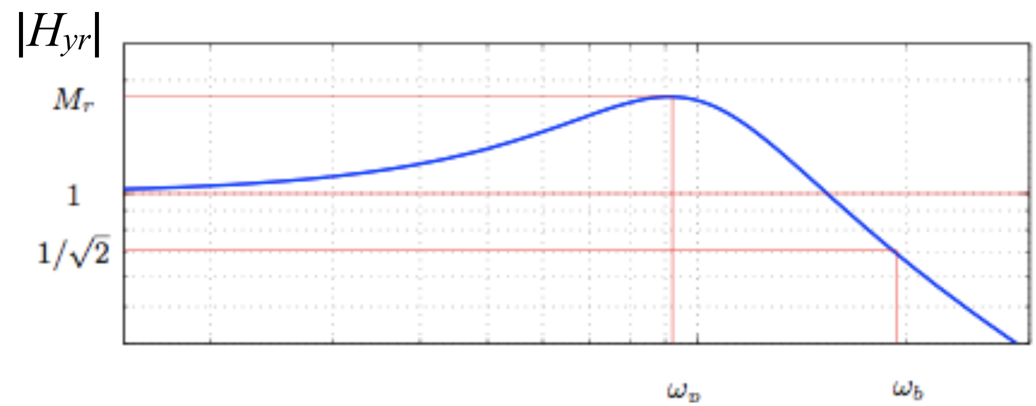
- Stability is always determined by 1/(1+*PC*) assuming stable process & controller
- Numerator determined by forward path between input and output

More generally: 6 primary transfer functions; simultaneous design of each

- Controller *C*(*s*) enters in multiple places ⇒ hard to understand tradeoffs

Frequency Domain Specifications





Specifications on the *open loop* transfer function (*L*)

- Gain crossover frequency, ω_{gc}, is the lowest frequency at which loop gain = 1
- Gain margin, g_m, is the amount the loop gain can be increased before instability
- Phase margin, φ_m, is amount of phase lag required to generate instability

Specifications on *closed loop* frequency response (eg H_{yr}, H_{yd}, etc)

- Resonant peak, M_r, is the largest value of the frequency response
- Peak frequency, ω_p , is the frequency where the maximum occurs
- Bandwidth, ω_b, is the frequency where the gain has decreased to $1/\sqrt{2}$

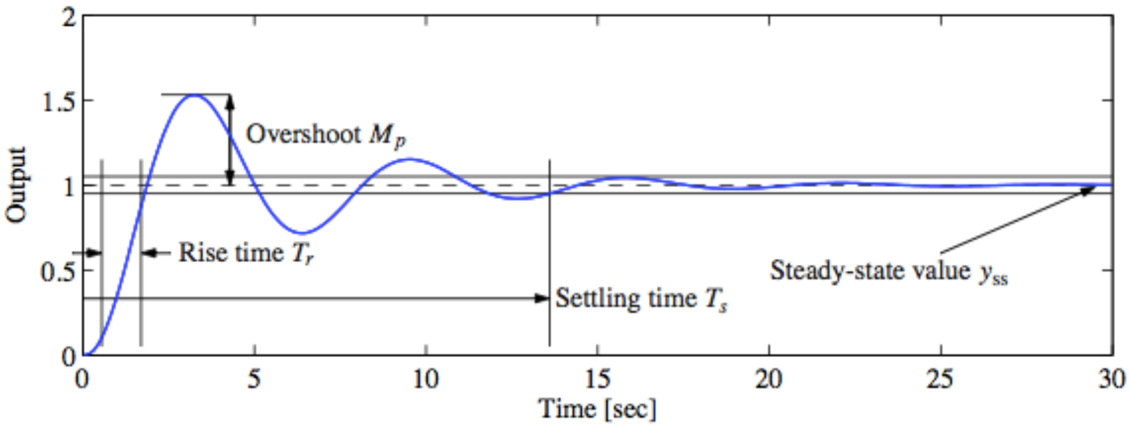
Basic idea: convert specs on closed loop to specs on open loop

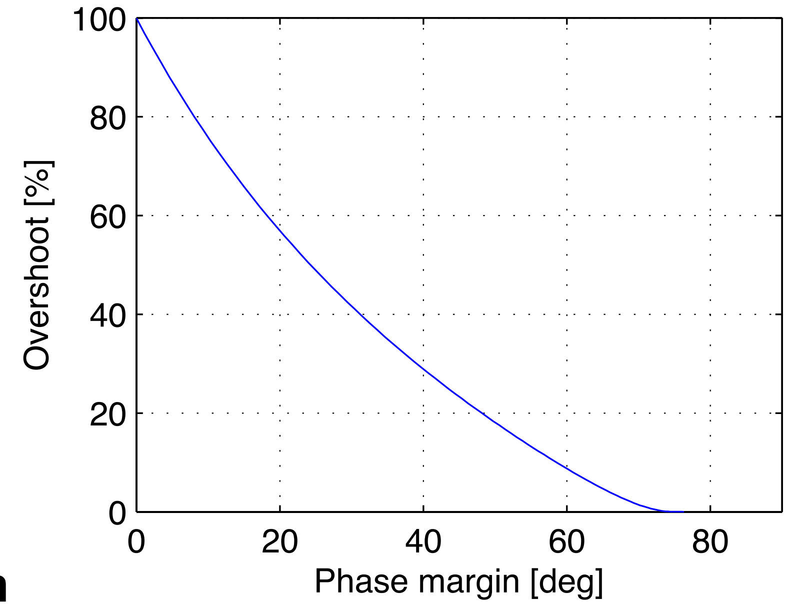
- Bandwidth ≈ value for which $|L| = 1$
- Resonant peak set by phase margin
- Keep L large to set $H_{yr} \approx 1$

$$H_{yr} = \frac{L}{1+L} \qquad H_{er} = \frac{1}{1+L}$$

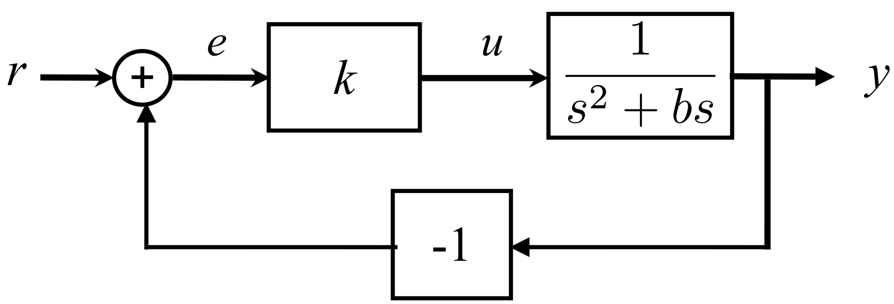
Time Domain Specs → Frequency Domain Specs

Time domain specifications



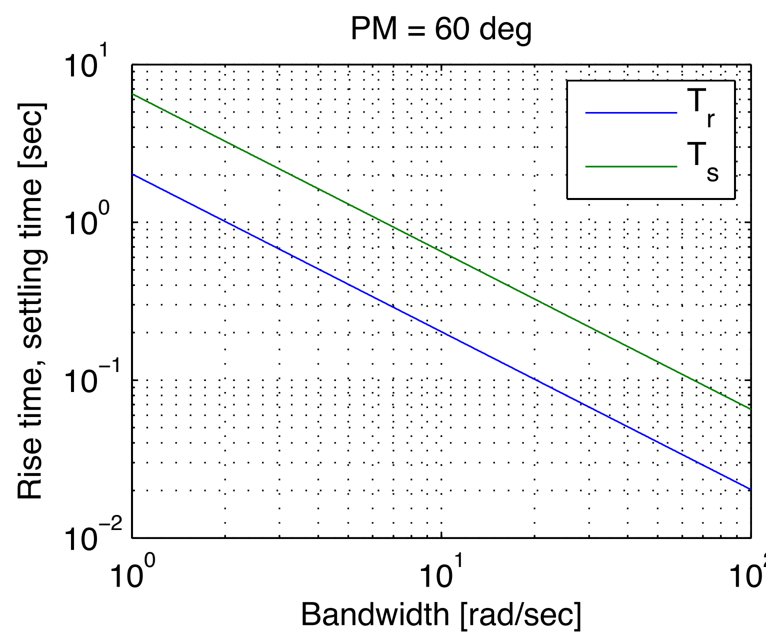


Map to frequency domain for second order system

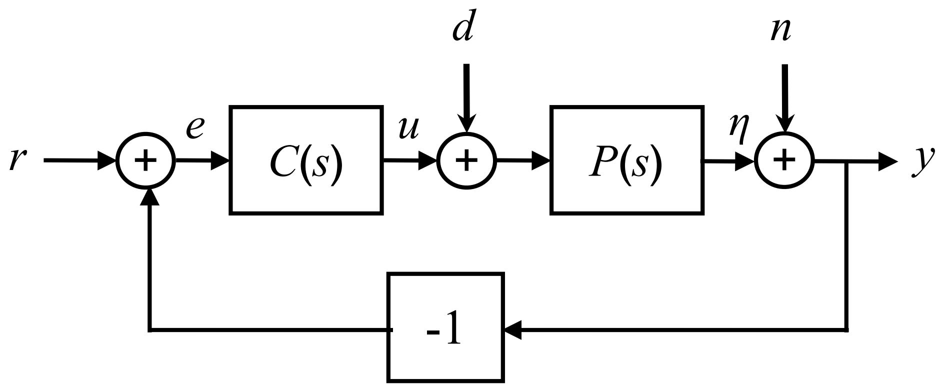


$$L(s) = \frac{k}{s^2 + bs} \qquad H_{yr} = \frac{k}{s^2 + bs + k}$$

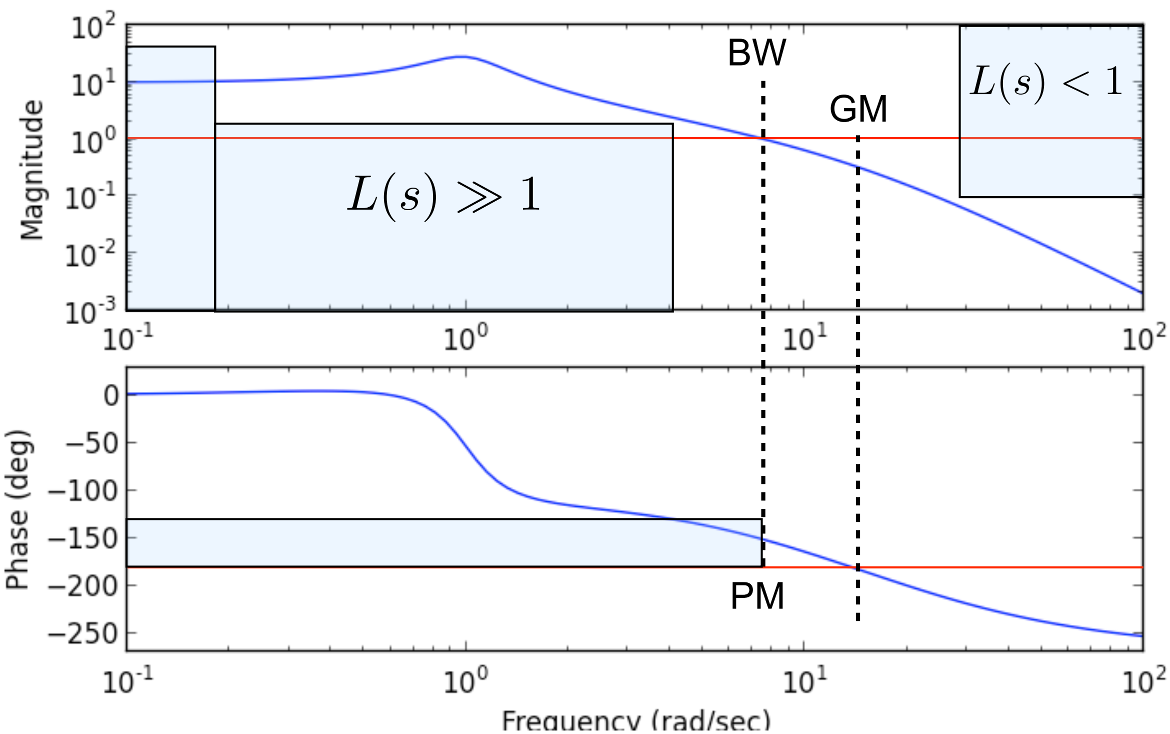
- Use properties of 2nd order systems (Table 7.1)



“Loop Shaping”: Design Loop Transfer Function



$$H_{er} = \frac{1}{1+L} \qquad H_{\eta n} = \frac{-L}{1+L}$$



Translate specs to “loop shape”

$$L(s) = P(s)C(s)$$

- Design C(s) to obey constraints

Typical loop constraints

- High gain at low frequency
  - Good tracking, disturbance rejection at low freqs
- Low gain at high frequency
  - Avoid amplifying noise
- Sufficiently high bandwidth
  - Good rise/settling time
- Shallow slope at crossover
  - Sufficient phase margin for robustness, low overshoot

Key constraint: slope of gain curve determines phase curve

- Can’t independently adjust
- Eg: slope at crossover sets PM

Example: Lead Compensation for Second Order System

System description

$$P(s) = \frac{p_1 p_2}{(s+p_1)(s+p_2)}$$

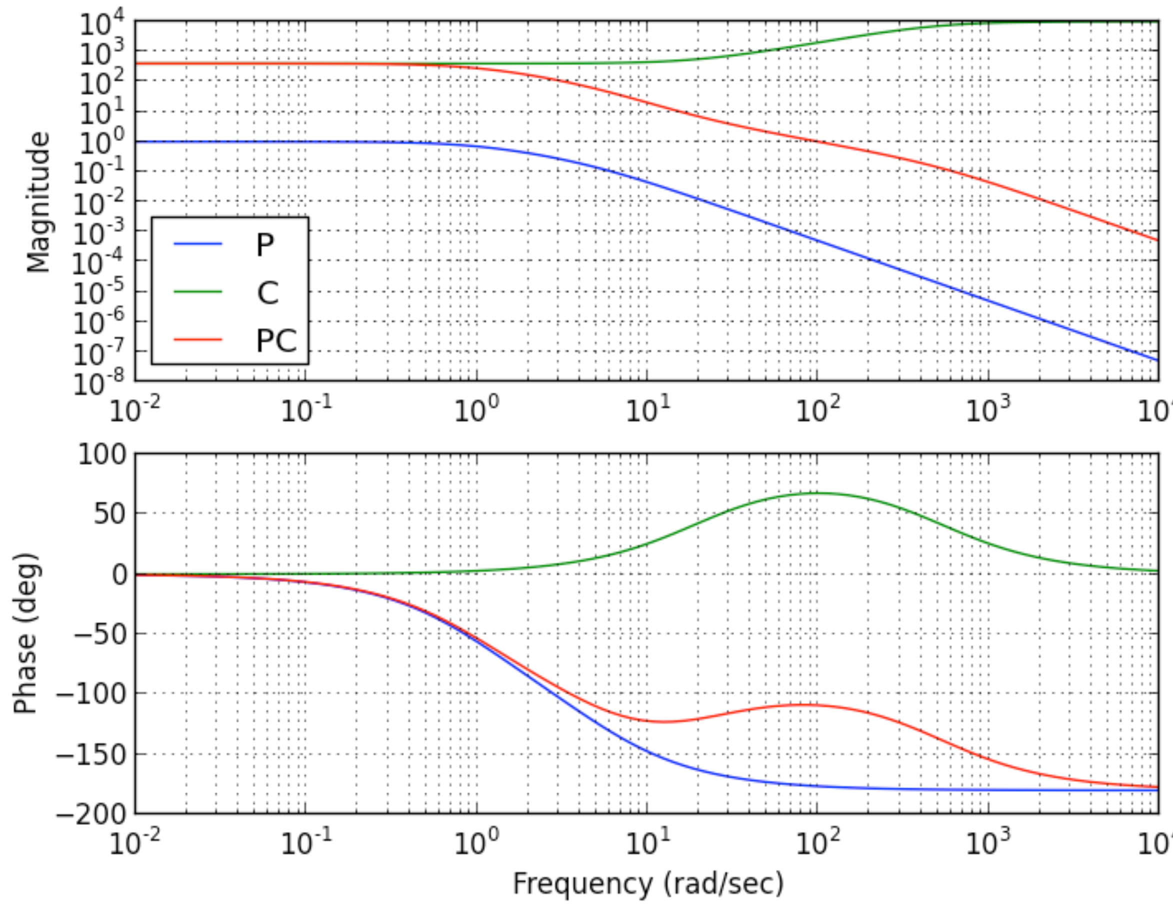
- Poles: p1 = 1, p2 = 5

Control specs

- Track constant reference with error < 1%
- Good tracking up to 100 rad/s (less than 10% error)
- Overshoot less than 10%
  - Gives PM of ~60 deg

Try a "lead" compensator

$$C(s) = K\frac{s+a}{s+b}$$



- Want gain crossover at approximately 100 rad/sec => center phase gain there
- Set zero frequency gain of controller to give small error => |L(0)| > 100
- a = 20, b = 500, K = 10,000 (gives |C(0)| = |L(0)| = 400)

Safety Check: Nyquist + Gang of 4

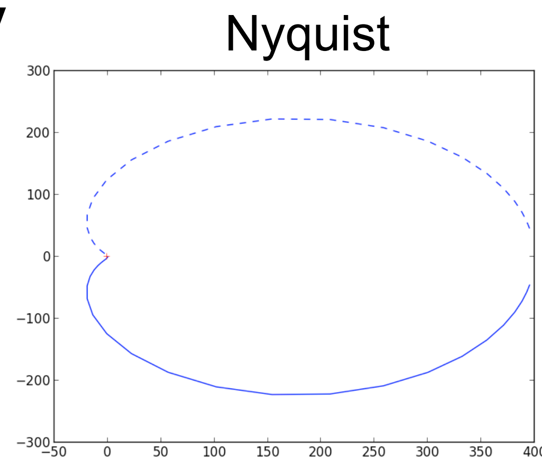
Nyquist verifies closed loop stability

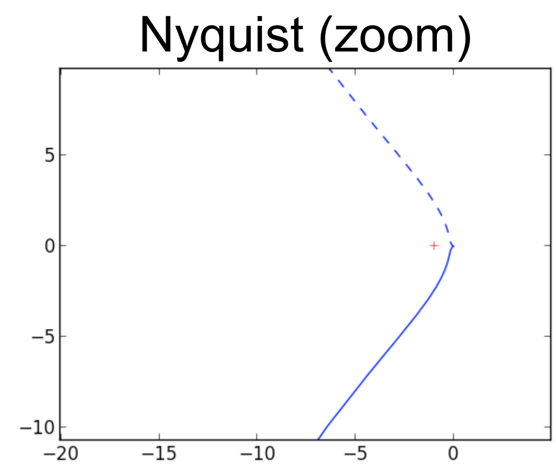
- Infinite GM; good phase margin

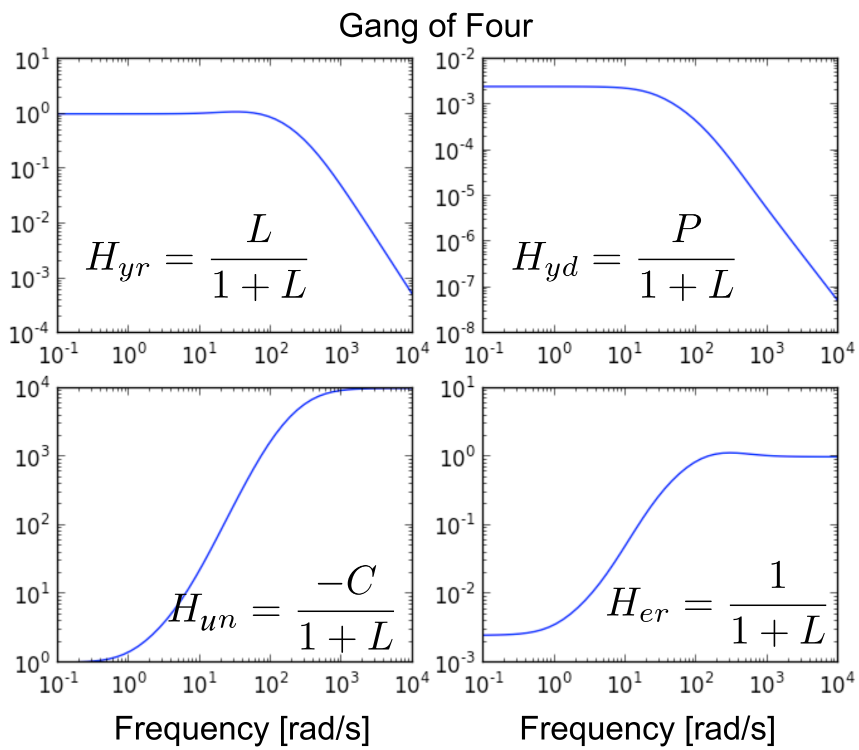
Gang of 4 shows high noise sens'y

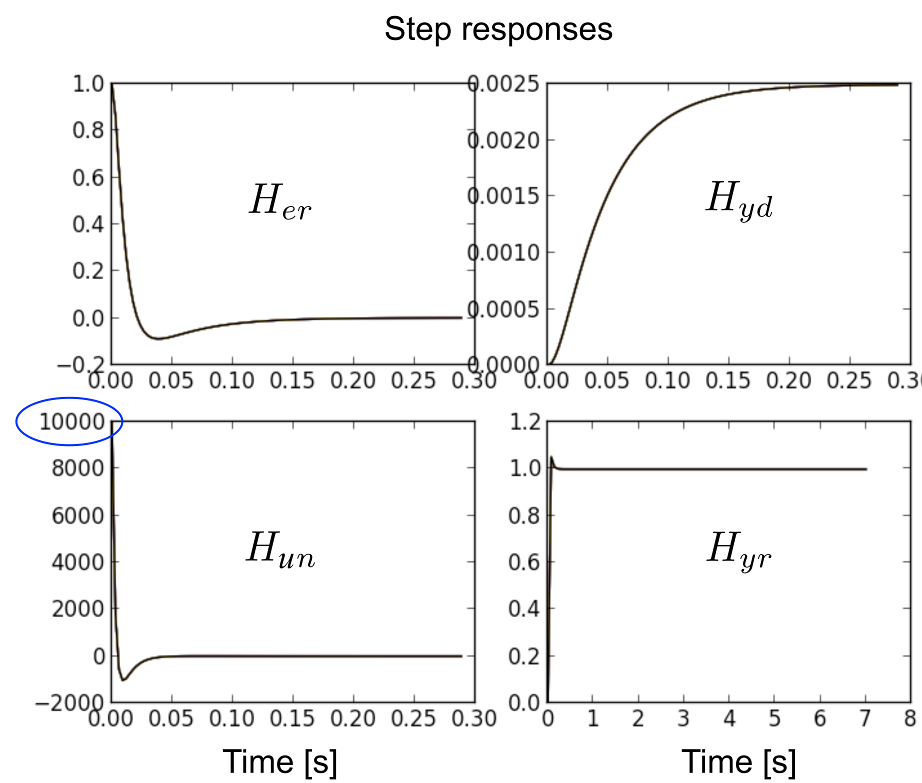
- Factor of 10,000 gain at high freq
- Step responses show similar sensitivity

Solution: high freq rolloff (L7-3)

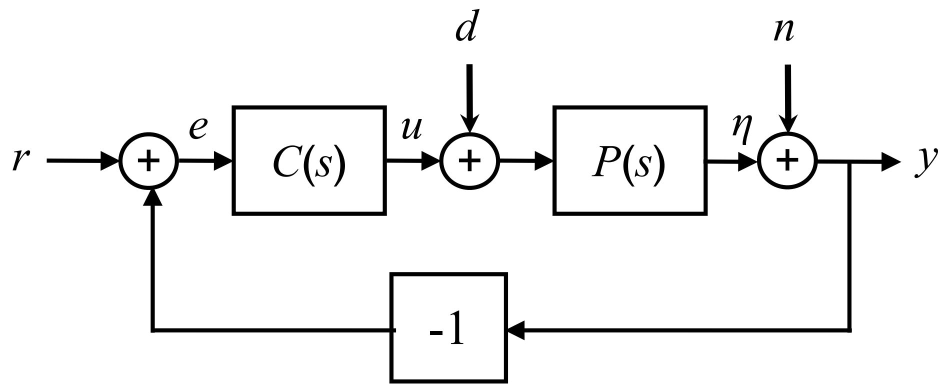








Algebraic Constraints on Performance



$$H_{er} = \frac{1}{1+PC} =: S$$ Sensitivity function

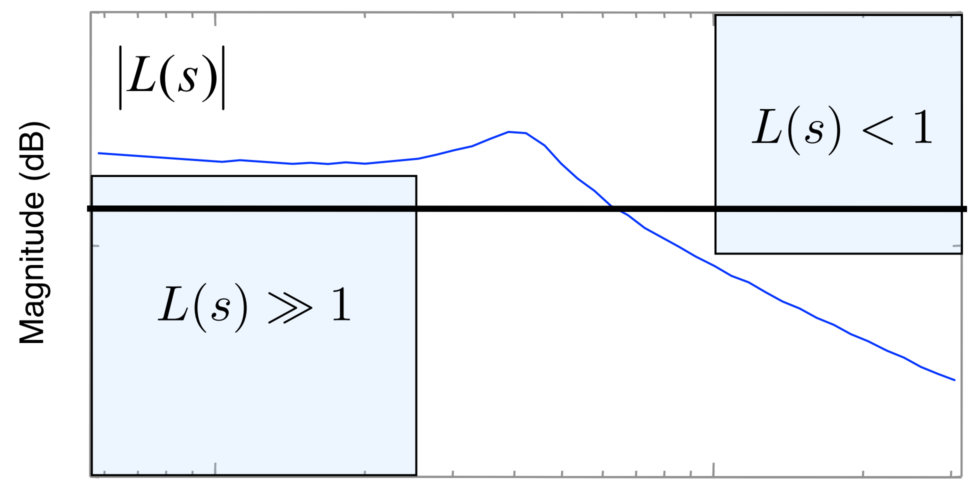
$$H_{yn} = \frac{PC}{1+PC} =: T$$ Complementary sensitivity function

Goal: keep S & T small

- S small ⇒ low tracking error
- T small ⇒ good noise rejection (and robustness [CDS 131/221])

Problem: S + T = 1

- Can't make both S & T small at the same frequency
- Solution: keep S small at low frequency and T small at high frequency
- Loop gain interpretation: keep L large at low frequency, and small at high frequency



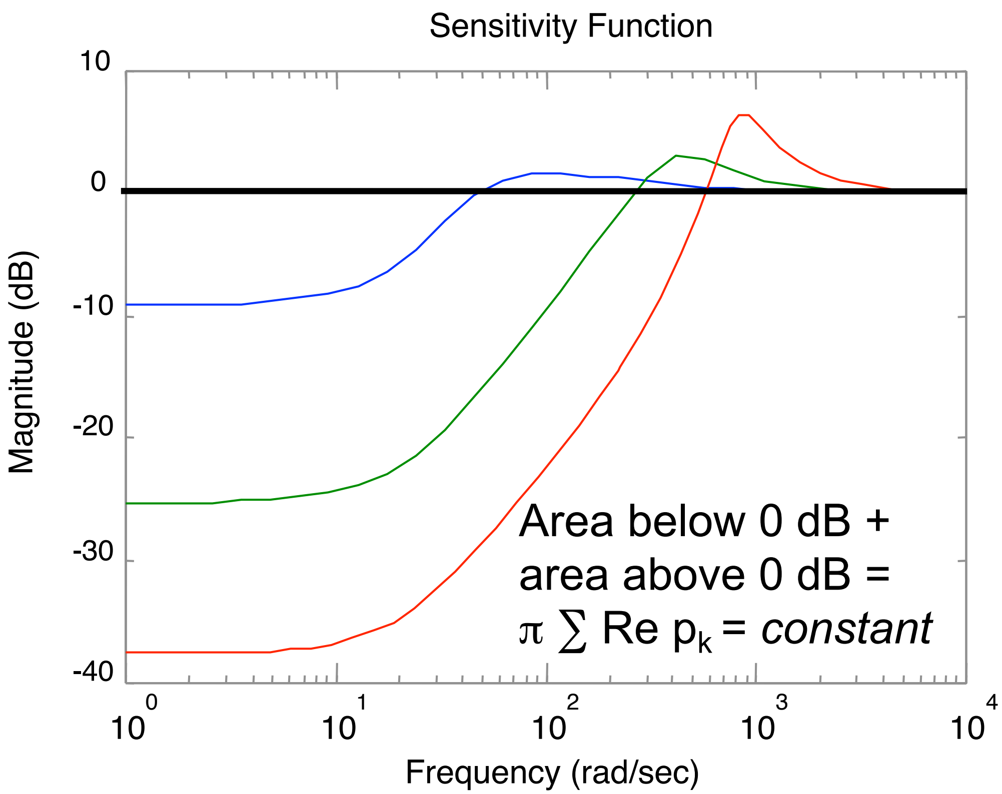
- Transition between large gain and small gain complicated by stability (phase margin)

Bode's Integral Formula and the Waterbed Effect

Bode's integral formula for S = 1/(1+PC) = 1/(1+L):

- Let pk be the unstable poles of L(s) and assume relative degree of L(s) ≥ 2
- Theorem: the area under the sensitivity function is a conserved quantity:

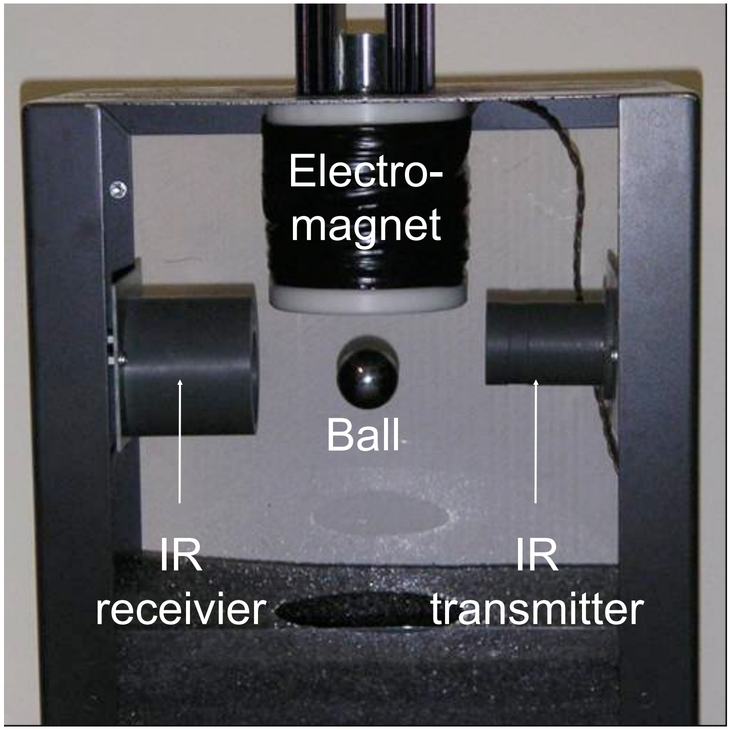
$$\int_0^\infty \log_e |S(j\omega)| d\omega = \int_0^\infty \log_e \frac{1}{|1 + L(j\omega)|} d\omega = \pi \sum \text{Re}\, p_k$$



Waterbed effect:

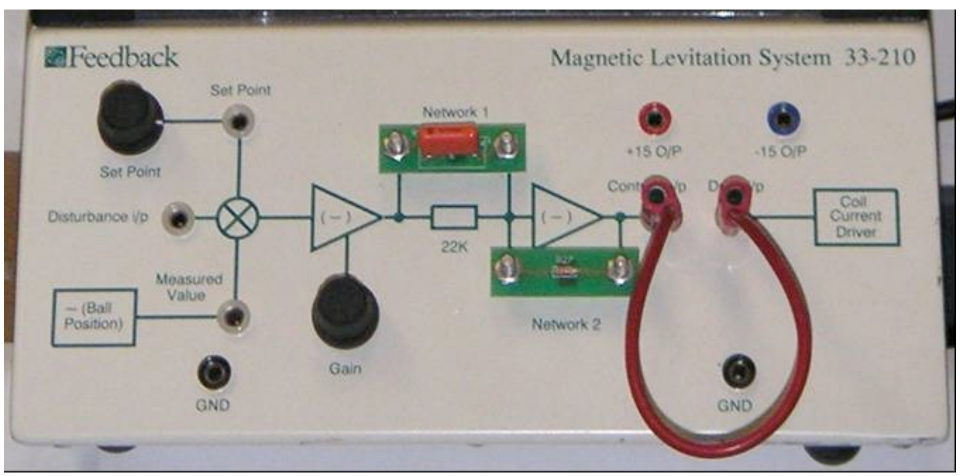
- Making sensitivity smaller over some frequency range requires *increase* in sensitivity someplace else
- Presence of RHP poles makes this effect worse
- Actuator bandwidth further limits what you can do
- Note: area formula is linear in ω; Bode plots are logarithmic

Example: Magnetic Levitation



System description

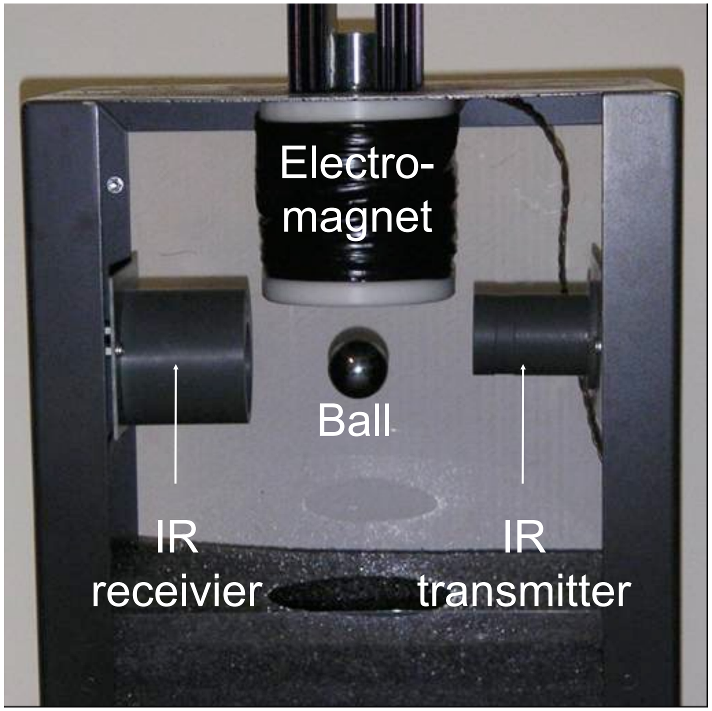
- Ball levitated by electromagnet
- Inputs: current thru electromagnet
- Outputs: position of ball (from IR sensor)
- States: $z, \ \dot{z}$
- Dynamics: F = ma, F = magnetic force generated by wire coil
- See MATLAB handout for details



Controller circuit

- Active R/C filter network
- Inputs: set point, disturbance, ball position
- States: currents and voltages
- Outputs: electromagnet current

Equations of Motion



Process: actuation, sensing, dynamics

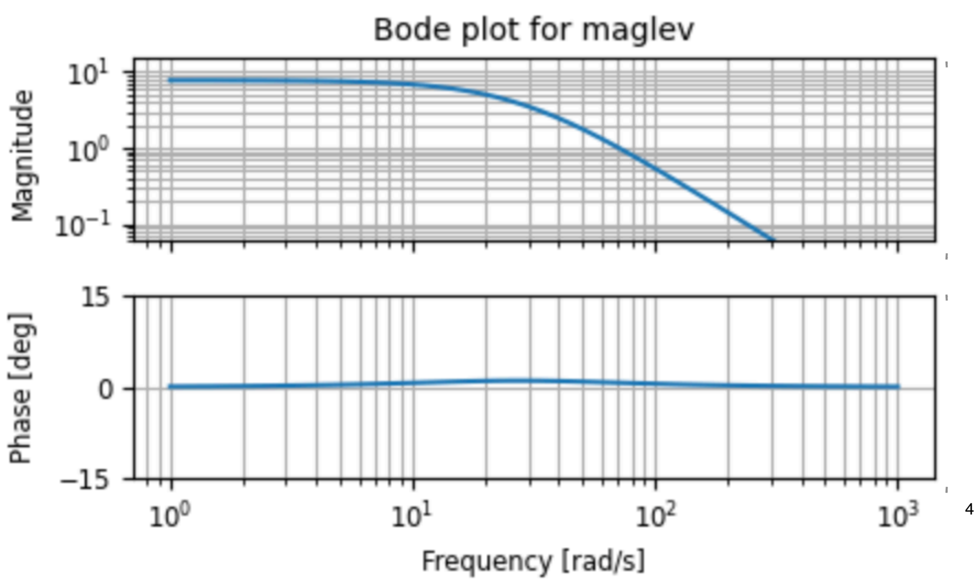
$$m\ddot{z} = mg - k_{\rm m}(k_{\rm A}u)^2/z^2 - c\dot{z}$$
$$v_{\rm ir} = k_T z + v_0$$

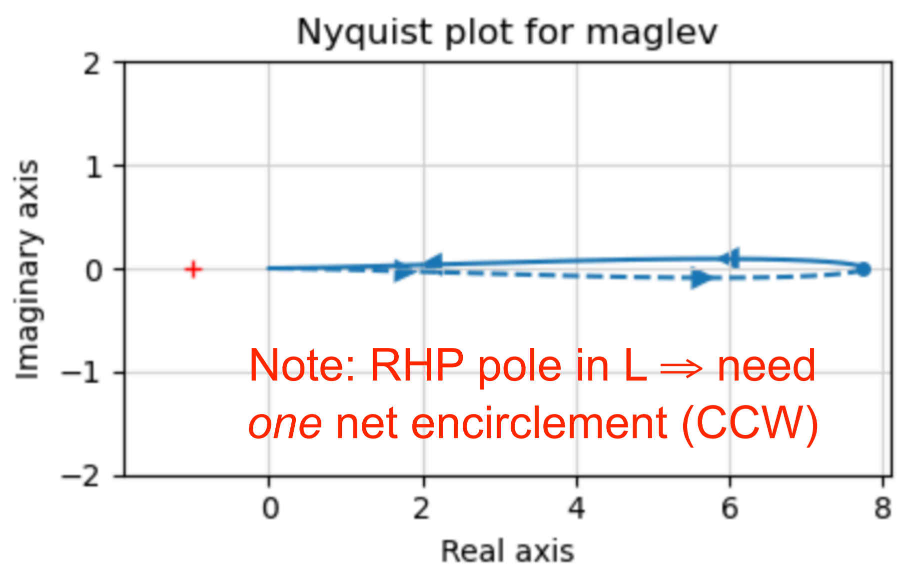
- u = current to electromagnet
- v_{ir} = voltage from IR sensor

Linearization:

$$P(s) = \frac{-k}{s^2 + cs - r^2} \qquad k, c, r > 0$$

- Poles at $s = \pm r \Rightarrow$ open loop unstable





Note: RHP pole in L ⇒ need *one* net encirclement (CCW)

Control Design

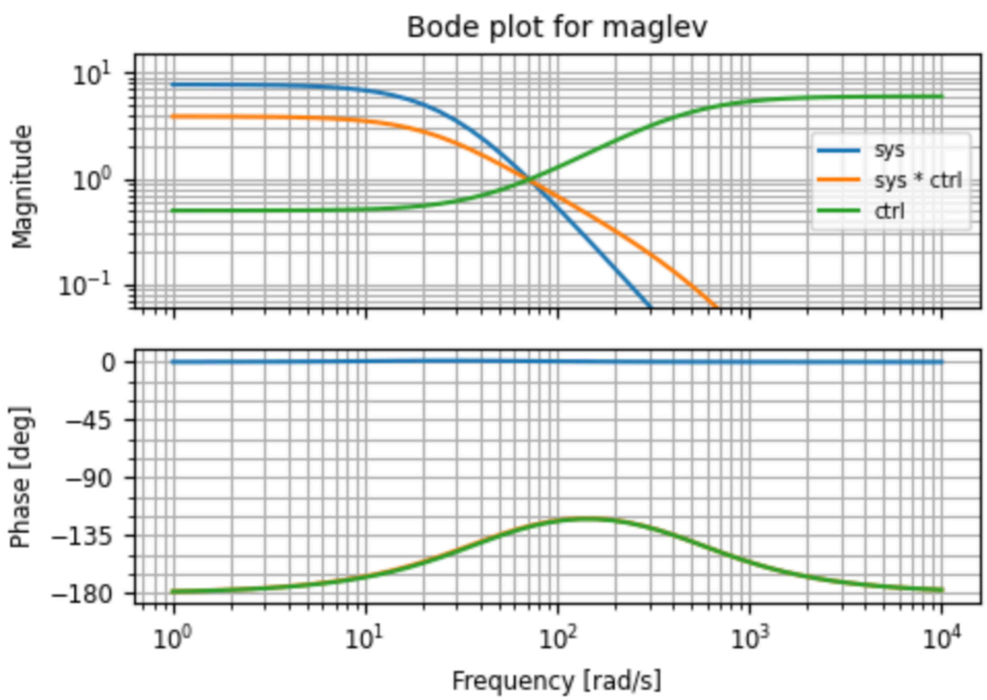
Need to create encirclement

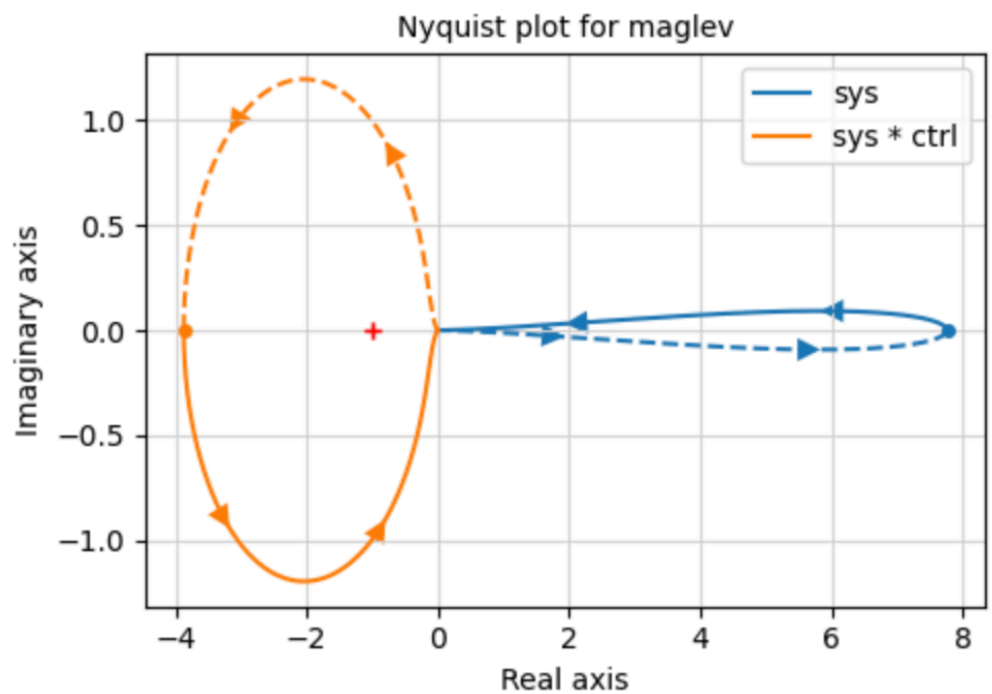
- Loop shaping is not useful here
- Flip gain to bring Nyquist plot over -1 point
- Insert phase to create CCW encirclement

Can accomplish using a lead compensator

- Produce phase lead at crossover
- Generates loop in Nyquist plot

$$C(s) = -k\frac{s+a}{s+b}$$





Performance Limits

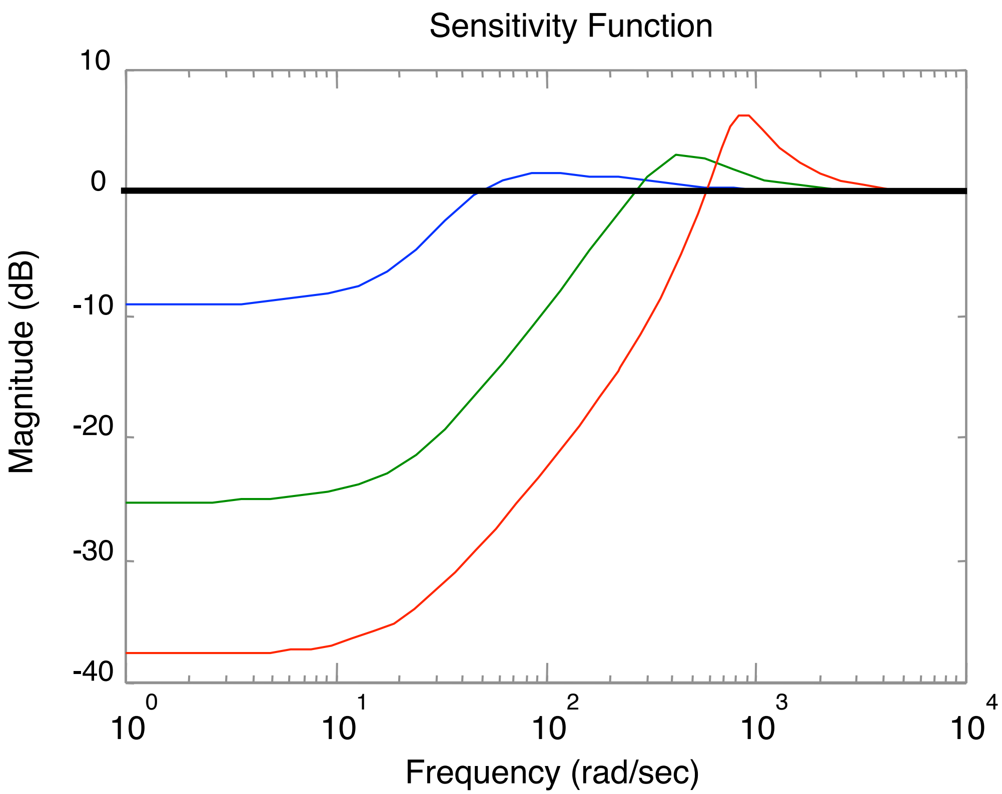
Nominal design gives low perf

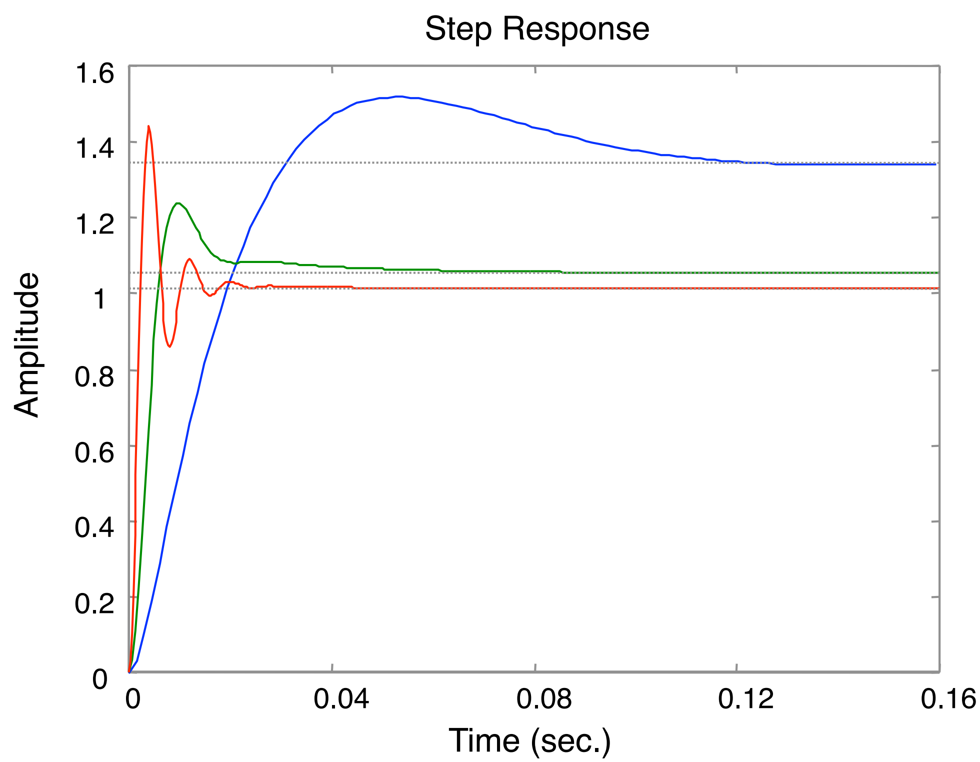
- Not enough gain at low frequency
- Try to adjust overall gain to improve low frequency response
- Works well at moderate gain, but notice waterbed effect

Bode integral limits improvement

$$\int_0^\infty \log|S(j\omega)|d\omega = \pi r$$

- Must increase sensitivity at some point



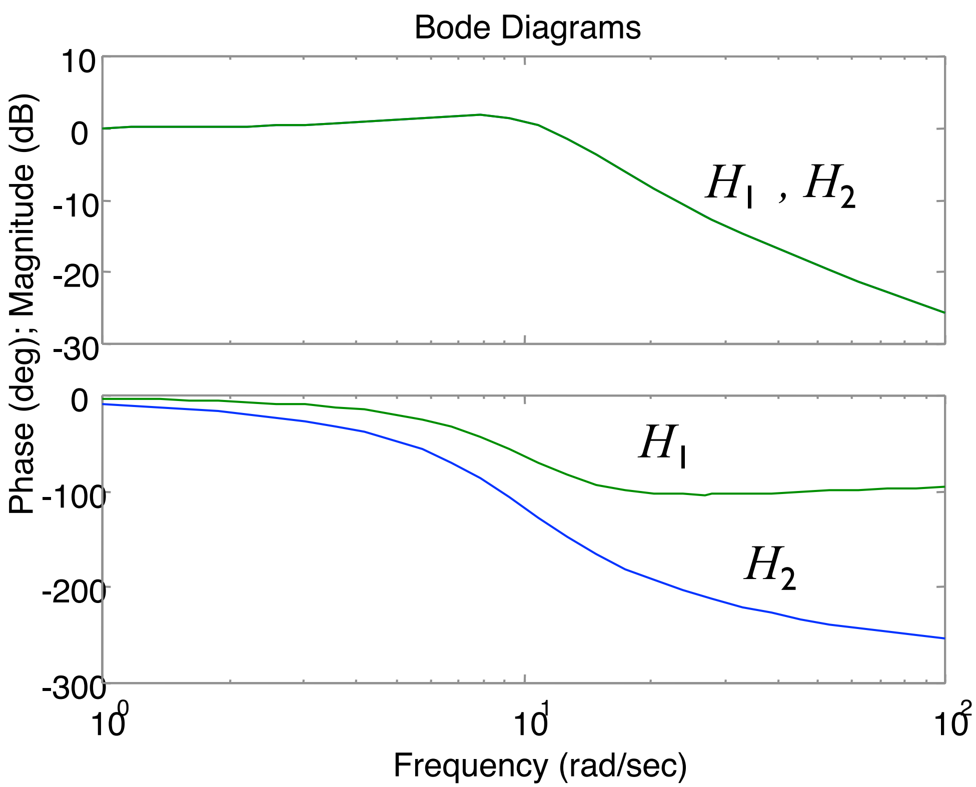


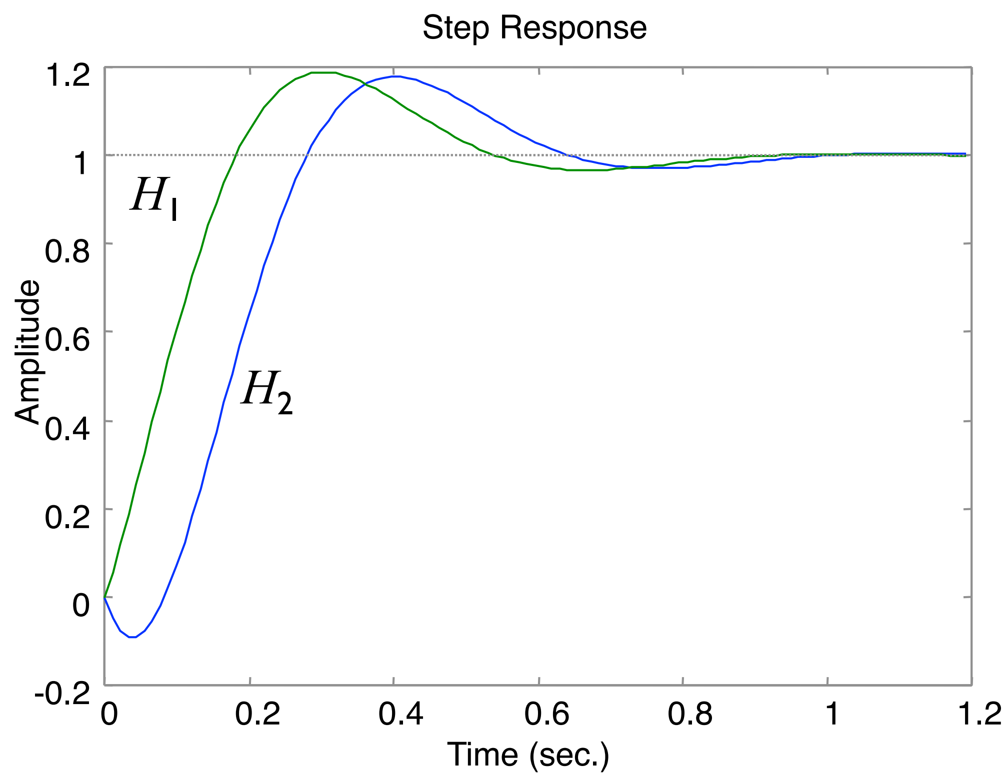
Right Half Plane Zeros

Right half plane zeros produce "non-minimum phase" behavior

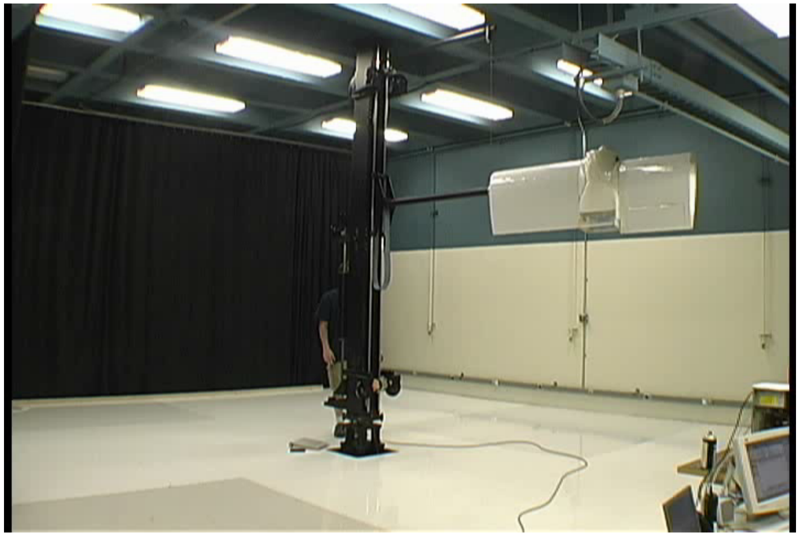
- Phase of frequency response has additional phase lag for given magnitude
- Can cause output to move opposite from input for a short period of time

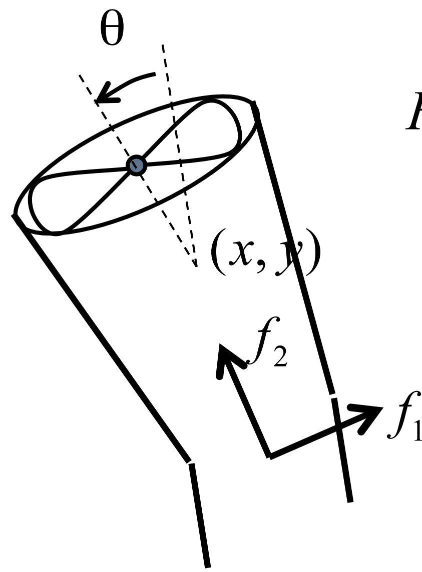
Example: $H_1(s) = \dfrac{s+a}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ **vs** $H_2(s) = \dfrac{s-a}{s^2 + 2\zeta\omega_n s + \omega_n^2}$





Example: Lateral Control of the Ducted Fan





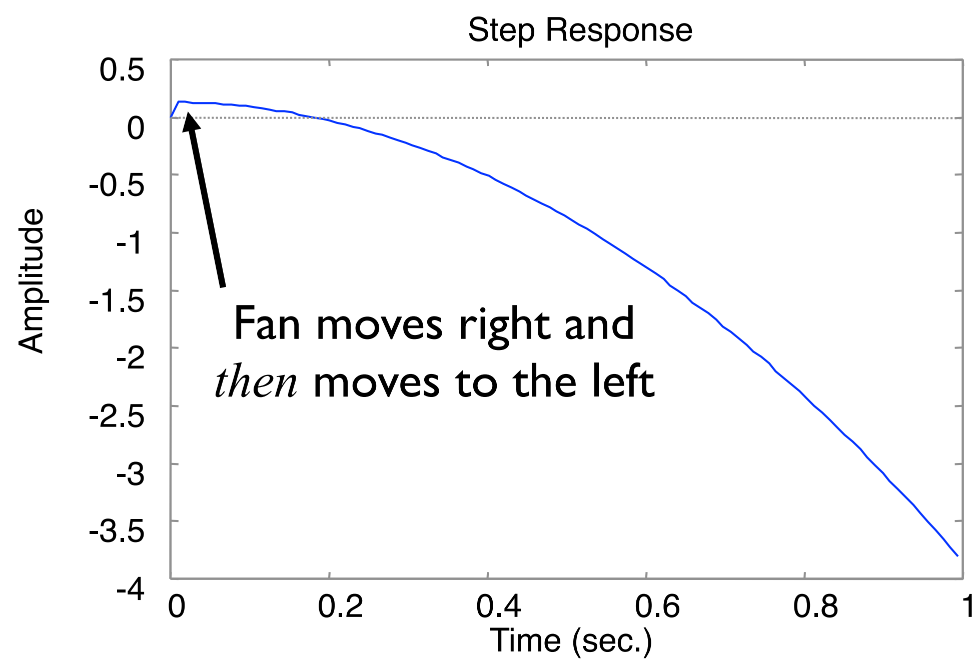
$$H_{xf_1}(s) = \frac{(s^2 - mgl)}{s^2(Js^2 + ds + mgl)}$$

- Poles: 0, 0, $-\sigma$ § $j\,\omega_d$
- Zeros: $\pm\sqrt{mgl}$

Source of non-minimum phase behavior

- To move left, need to make θ > 0
- To generate positive θ, need f1 > 0
- Positive f1 causes fan to move right initially
- Fan starts to move left after short time (as fan rotates)

Will show on Wed that RHP zeros impose limits on achievable performance



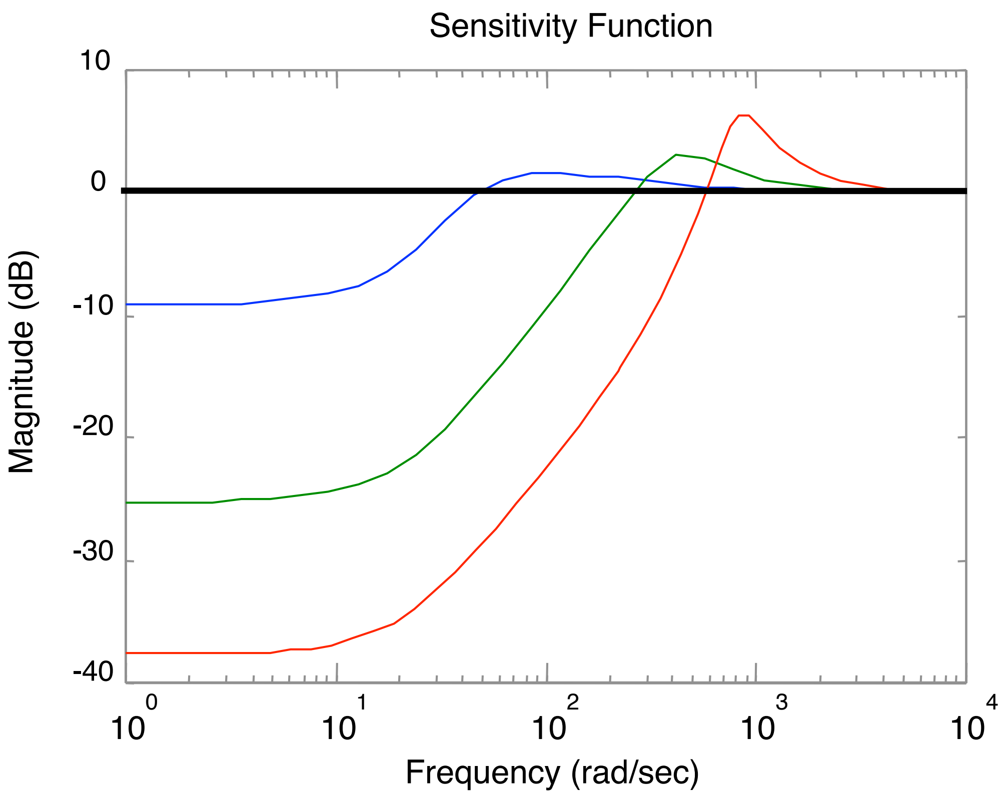
Summary: Limits of Performance

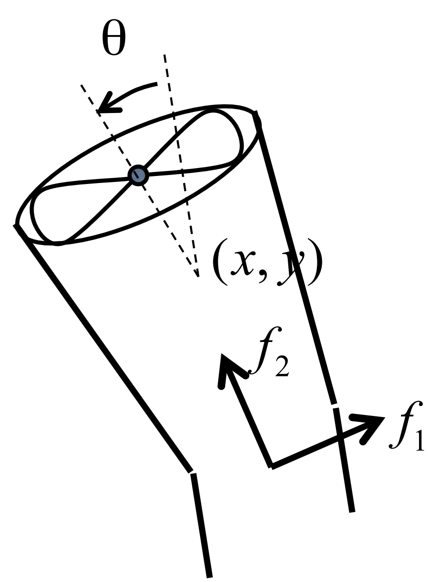
Many limits to performance

- Algebraic: S + T = 1
- RHP poles: Bode integral formula
- RHP zeros: Waterbed effect on peak of S

Main message: try to avoid RHP poles and zeros whenever possible (eg, re-design)

$$\int_0^\infty \log_e |S(j\omega)| d\omega = \int_0^\infty \log_e \frac{1}{|1 + L(j\omega)|} d\omega = \pi \sum \text{Re}\, p_k$$





Announcements

Final exam

- Seniors/grad students: 7 Jun (Fri) @ 2 pm
  - Fr, So, Jr can take exam early
- Fr, So, Jr: 12 Jun (Wed) @ 2 pm
- In person, closed book, 1 hour
  - T/F, multiple choice, short answer
  - Similar to final problem on HW

Date changes for final weeks of class

- 27 May: Memorial day => 2 day shift
- HW8: due date 29 May (W) → 31 May (F)
  - W8 office hours: Wed, 3-4 and Thu, 4-5
- W9: 29 May (W), 31 May (F), 3 Jun (M)
- W9 office hours: Wed, 3-4 and Thu, 4-5
  - HW #9 (Fr, So, Jr): due 7 Jun (Fri)
  - HW 9 or questions about class
- Review for final on 5 Jun (Wed)
  - No class on 7 Jun (Fri) [final for Sr/Gr]
- Final OH: 10 Jun (M), 3-4, 11 Jun (Tu), 4-5