I. Basic feedback loop

A. \( x = Ax + Bu \quad u = e^t \quad y = Cx + Du \quad G(s) = C(sI - A)^{-1}B + D \quad G(s) = \frac{n(s)}{d(s)} \)

B. \( r \rightarrow e \rightarrow G(s) \rightarrow u \rightarrow P(s) \rightarrow y \)

\( G_{yr}(s) = \frac{PC}{1 + PC} \)

\( G_{un}(s) = \frac{C}{1 + PC} \quad \text{etc} \)

C. Control design: \( P \) given ⇒ to get \( y \) to track \( r \), make \( C \) large ⇒ \( G_{yr}(s) \approx 1 \)

But if \( P \) is small (eg at high freq) then \( G_{un} \) could be large ⇒ amplify noise ⇒ tradeoffs (more next week)

II. Stability: Q: given \( P \) and \( C \), when is closed loop stable?

A. Algebra: compute \( \frac{1}{1+PC} = \frac{1}{1+C} = \frac{dp(s)dc(s)}{dpdc + npnc} \quad \text{look at zeroes} \)

B. Nyquist criterion: \( Z = N + P \)

\( \text{LHP} \Rightarrow \text{stable, RHP} \Rightarrow \text{unstable} \)

RHP poles of closed loop ⇒ RHP poles of \( L(s) = PC \)

encirclements of \( -1 \)

C. Intuition: go around the loop with gain \( > 1 \) @ \( 180^\circ \) ⇒ signal grows

- For example above, decrease controller gain ⇒ stable (but low perf?)
- Be careful if \( P(s) \) (or \( C(s) \)) is unstable: need CCW encirclement

0. Gain, phase, and stability margins (see textbook for details)
Goals:
• Show how to compute closed loop stability from open loop properties
• Describe the Nyquist stability criterion for stability of feedback systems
• Define gain and phase margin and determine it from Nyquist and Bode plots

Reading:
• Åström and Murray, Feedback Systems, Ch 10
Review From Monday

\[ u = A \sin(\omega t) \]

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]
\[ x(0) = 0 \]

\[ y_{ss} = A \cdot |G(i\omega)| \times \sin(\omega t + \arg G(i\omega)) \]

\[ G(s) = C(sI - A)^{-1} B + D \]

\[ G_{y_2u_1} = G_{y_2u_2} G_{y_1u_1} = \frac{n_1 n_2}{d_1 d_2} \]
Closed Loop Stability

Q: how do open loop dynamics affect the closed loop stability?

- Given open loop transfer function $C(s)P(s)$ determine when system is stable

Brute force answer: compute poles closed loop transfer function

$$H_{yr} = \frac{PC}{1 + PC} = \frac{n_p n_c}{d_p d_c + n_p n_c}$$

- Poles of $H_{yr}$ = zeros of $1 + PC$
- Easy to compute, but not so good for design

Alternative: look for conditions on $PC$ that lead to instability

- Example: if $PC(s) = -1$ for some $s = i\omega$, then system is not asymptotically stable
- Condition on $PC$ is much nicer because we can design $PC(s)$ by choice of $C(s)$
- However, checking $PC(s) = -1$ is not enough; need more sophisticated check
Game Plan: Frequency Domain Design

Goal: figure out how to design $C(s)$ so that $1 + C(s)P(s)$ is stable and we get good performance

$$H_{yr} = \frac{PC}{1 + PC}$$

- Poles of $H_{yr} = $ zeros of $1 + PC$
- Would also like to “shape” $H_{yr}$ to specify performance at different frequencies

- Low frequency range:
  $PC \approx 1 \Rightarrow \frac{PC}{1 + PC} \approx 1$ (good tracking)
- Bandwidth: frequency at which closed loop gain = $\frac{1}{2}$
  $\Rightarrow$ open loop gain $\approx 1$

- Idea: use $C(s)$ to shape $PC$
- Need tools to analyze stability and performance for closed loop given $PC$
Nyquist Criterion

Determine stability from (open) loop transfer function, \( L(s) = P(s)C(s) \).
- Use “principle of the argument” from complex variable theory (see reading)

**Thm (Nyquist).** Consider the Nyquist plot for loop transfer function \( L(s) \). Let
- \( P \) \# RHP poles of \( L(s) \)
- \( N \) \# clockwise encirclements of -1
- \( Z \) \# RHP zeros of \( 1 + L(s) \)

Then
\[
Z = N + P
\]
Simple Interpretation of Nyquist

Basic idea: avoid positive feedback
- If $L(s)$ has $180^\circ$ phase (or greater) and gain greater than 1, then signals are amplified around loop
- Use when phase is monotonic
- General case requires Nyquist

Can generate Nyquist plot from Bode plot + reflection around real axis

\[
\begin{align*}
&d \\
&\downarrow \\
&e \\
&\downarrow \\
&r\rightarrow C(s) \rightarrow + \rightarrow u \rightarrow P(s) \rightarrow y \\
&\downarrow \\
&-1
\end{align*}
\]

Bode Diagrams

Nyquist Diagrams

\[\text{ambode(sys) [or bode(sys) in dB]}\]

\[\text{amnyquist(sys)}\]
Example: Proportional + Integral* speed controller

Example: Proportional + Integral* speed controller

Nyquist Diagrams
From: U(1)

\[
\begin{align*}
C(s) &= K_p + \frac{K_i}{s + 0.01} \\
P(s) &= \frac{1/m}{s + b/m} \times \frac{r}{s + a}
\end{align*}
\]

Remarks
- N = 0, P = 0 \Rightarrow Z = 0 (stable)
- Need to zoom in to make sure there are no net encirclements
- Note that we don’t have to compute closed loop response

* slightly modified; more on the design of this compensator in next week’s lecture
More complicated systems

What happens when open loop plant has RHP poles?

- $1 + PC$ has singularities inside $D$ contour $\Rightarrow$ these must be taken into account

\[
L(s) = \frac{s + 1}{s - 0.5} \times \frac{1}{s^2 + s + 1}
\]

unstable pole

\[
\frac{1}{1 + L} = \frac{s + 1}{(s + 0.35)(s + 0.07 + 1.2j)(s + 0.07 - 1.2j)}
\]

$N = -1, P = 1 \Rightarrow Z = N+P = 0$ (stable)
Comments and cautions

Why is the Nyquist plot useful?
- Old answer: easy way to compute stability (before computers and MATLAB)
- Real answer: gives insight into stability and robustness; very useful for reasoning about stability

Nyquist plots for systems with poles on the $j\omega$ axis
- chose contour to avoid poles on axis
- need to carefully compute Nyquist plot at these points
- evaluate $H(\epsilon+0i)$ to determine direction

Cautions with using MATLAB
- MATLAB doesn’t generate portion of plot for poles on imaginary axis
- These must be drawn in by hand (make sure to get the orientation right!)
- (Python-control toolbox does a much better job of plotting, but still need to take care)
Robust stability: gain and phase margins

Nyquist plot tells us if closed loop is stable, but not how stable

Gain margin
- How much we can modify the loop gain and still have the system be stable
- Determined by the location where the loop transfer function crosses 180° phase

Phase margin
- How much we can add “phase delay” and still have the system be stable
- Determined by the phase at which the loop transfer function has unity gain

Bode plot interpretation
- Look for gain = 1, 180° phase crossings
- MATLAB: margin(sys)
Example: cruise control

Effect of additional sensor dynamics

- New speedometer has pole at $s = 10$ (very fast); problems develop in the field
- What’s the problem? A: insufficient phase margin in original design (not robust)

$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s + a}$$

$$C(s) = K_p + \frac{K_i}{s + 0.01}$$

$$G(s) = \frac{10}{s + 10}$$
Summary: Loop Analysis of Feedback Systems

- Nyquist criteria for loop stability
- Gain, phase margin for robustness

Thm (Nyquist).

\[ P \] # RHP poles of \( L(s) \)
\[ N \] # CW encirclements
\[ Z \] # RHP zeros

\[ Z = N + P \]