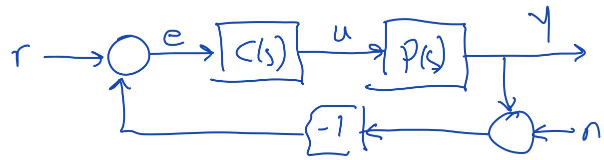
CDS 110, L7-2: Loop Analysis (Nyquist)

RMM 14 May 2024

I. Basic feedback loop

A. $\dot{x} = Ax + Bu$, $y = Cx + Du$ $\quad u = e^{st} \quad y = G(s)e^{st} +$ ~~transients~~ ($\to 0$) $\quad G(s) = C(sI-A)^{-1}B + D = n(s)/d(s)$

B. 

$$G_{yr}(s) = \frac{PC}{1+PC}$$

$$G_{un}(s) = \frac{C}{1+PC} \quad \text{etc}$$

C. Control design: P given ⇒ to get y to track r, make C large ⇒ $G_{yr}(s) \approx 1$

<u>But</u> if P is small (eg at high freq) then G_{un} could be large ⇒ amplify noise ⇒ tradeoffs (more next week)

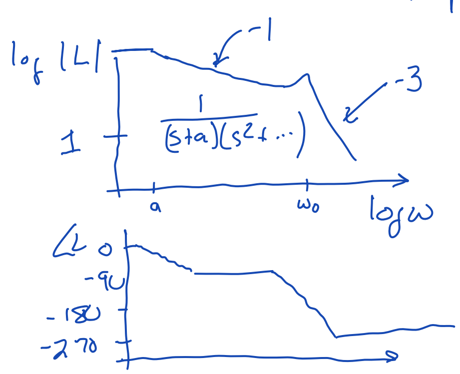
II. Stability: Q: given P & C, when is closed loop <u>stable</u>

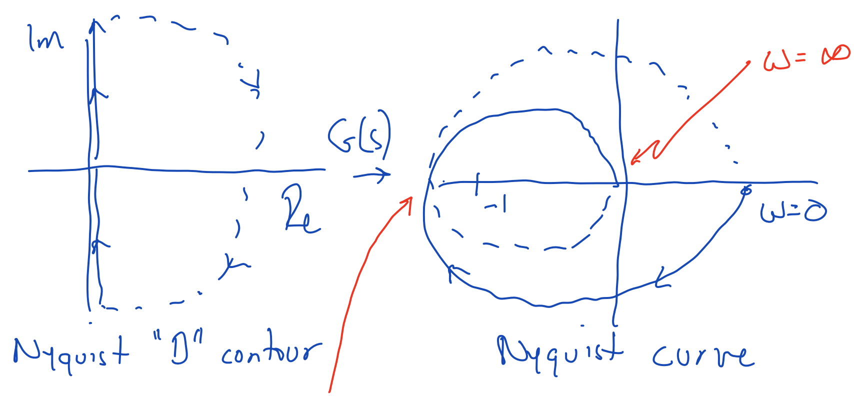
A. Algebra: compute $\frac{1}{1+PC} = \frac{1}{1+L} = \frac{d_p(s)d_c(s)}{d_p d_c + n_p n_c}$ ← look at zeros

LHP ⇒ stable, RHP ⇒ unstable

B. Nyquist criterion: $Z = N + P$

- Z: RHP poles of closed loop
- N: encirclements of -1
- P: RHP poles of $L(s) = PC$





C. Intuition: go around the loop w/ gain > 1 @ 180° ⇒ signal grows

- For example above, decrease controller gain ⇒ stable (but low perf?)
- Be careful if P(s) (or C(s)) is unstable: <u>need</u> CCW encirclement

D. Gain, phase, and stability margins (see textbook for details)

CDS 101/110: Lecture 7-2
Loop Analysis of Feedback Systems

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9 November 2015

Goals:

- Show how to compute closed loop stability from open loop properties
- Describe the Nyquist stability criterion for stability of feedback systems
- Define gain and phase margin and determine it from Nyquist and Bode plots

Reading:

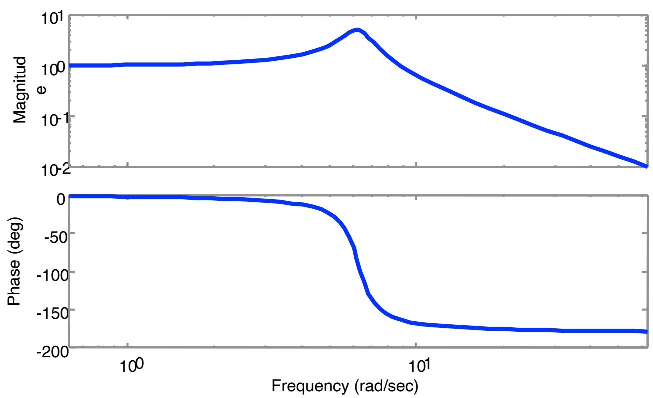
- Åström and Murray, Feedback Systems, Ch 10

Review From Monday

$$u = A\sin(\omega t)$$

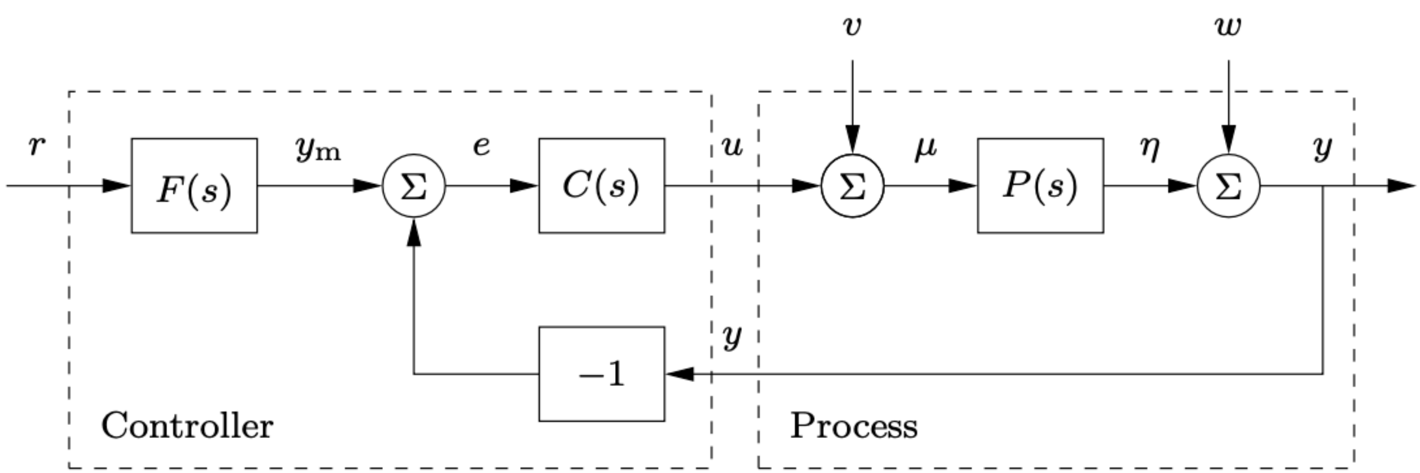
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$x(0) = 0$$

$$y_{ss} = A \cdot |G(i\omega)| \times \sin(\omega t + \arg G(i\omega))$$

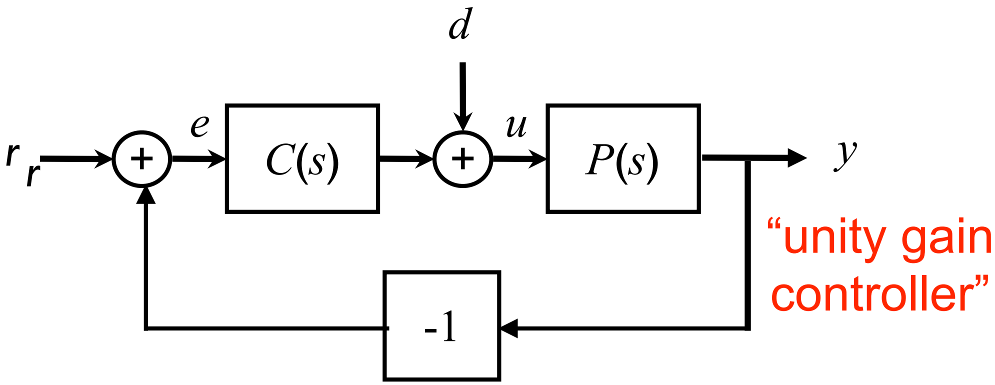


$$G(s) = C(sI - A)^{-1}B + D$$

$$G_{y_2u_1} = G_{y_2u_2}G_{y_1u_1} = \frac{n_1n_2}{d_1d_2}$$



Closed Loop Stability



Q: how do open loop dynamics affect the closed loop stability?

- Given open loop transfer function $C(s)P(s)$ determine when system is stable

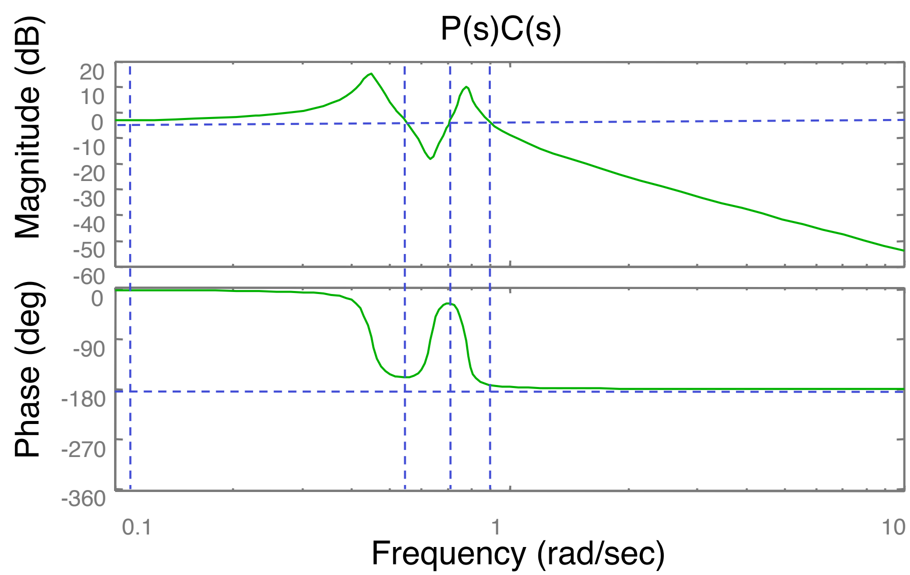
Brute force answer: compute poles closed loop transfer function

$$H_{yr} = \frac{PC}{1+PC} = \frac{n_p n_c}{d_p d_c + n_p n_c}$$

- Poles of H_{yr} = zeros of 1 + PC
- Easy to compute, but not so good for design

Alternative: look for conditions on *PC* that lead to instability

- Example: if *PC*(*s*) = -1 for some *s* = *i*ω, then system is *not* asymptotically stable
- Condition on *PC* is much nicer because we can *design PC*(*s*) by choice of *C*(*s*)
- However, checking *PC*(*s*) = -1 is not enough; need more sophisticated check

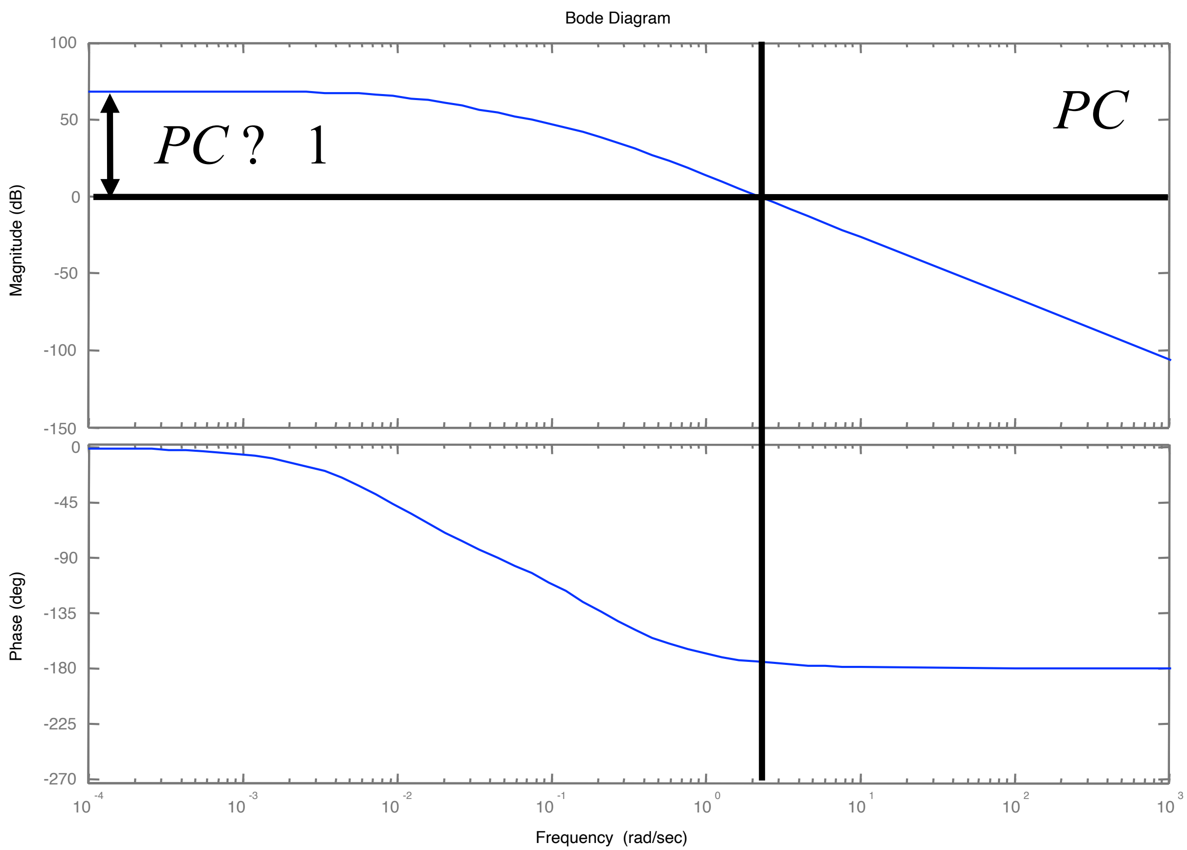


Game Plan: Frequency Domain Design

Goal: figure out how to *design* *C*(*s*) so that 1+*C*(*s*)*P*(*s*) is stable *and* we get good performance

$$H_{yr} = \frac{PC}{1+PC}$$

- Poles of H_{yr} = zeros of 1 + *PC*
- Would also like to "shape" H_{yr} to specify performance at different frequencies

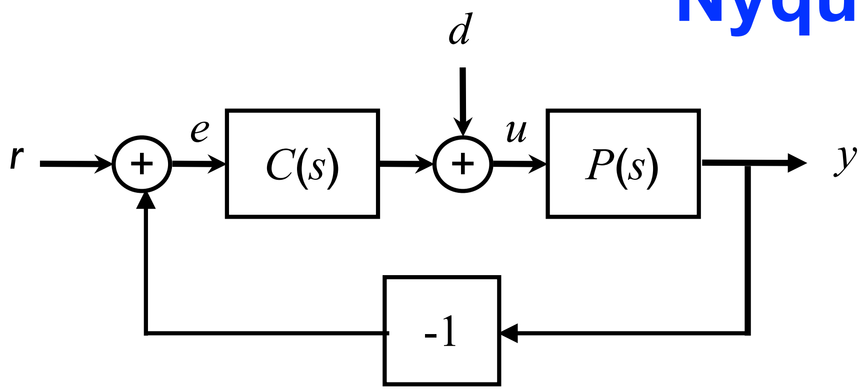


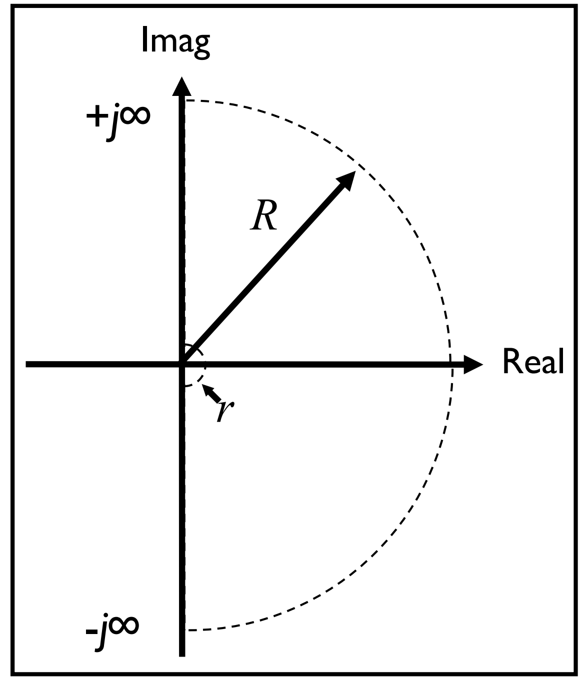
- Low frequency range:

$$PC\ ?\ 1 \Rightarrow \frac{PC}{1+PC} \approx 1$$

  (good tracking)
- Bandwidth: frequency at which closed loop gain = ½ ⇒ open loop gain ≈ 1
- Idea: use *C*(*s*) to *shape PC* (under certain constraints)
- Need tools to analyze stability and performance for closed loop given *PC*

Nyquist Criterion





- Nyquist "D" contour
- Take limit as $r \to 0$, $R \to \infty$
- Trace from –1 to +1 along imaginary axis

Determine stability from (open) loop transfer function, L(s) = P(s)C(s).

- Use "principle of the argument" from complex variable theory (see reading)

Thm (Nyquist). Consider the Nyquist plot for loop transfer function $L(s)$. Let

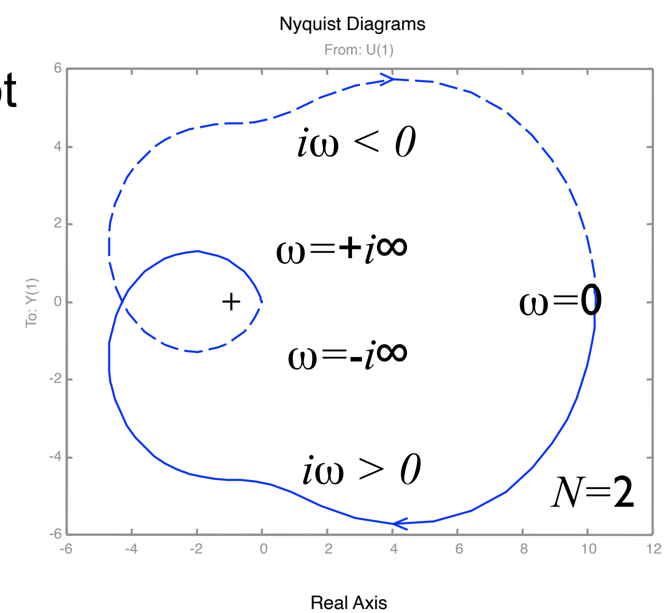
P # RHP poles of $L(s)$

N # clockwise encirclements of -1

Z # RHP zeros of $1 + L(s)$

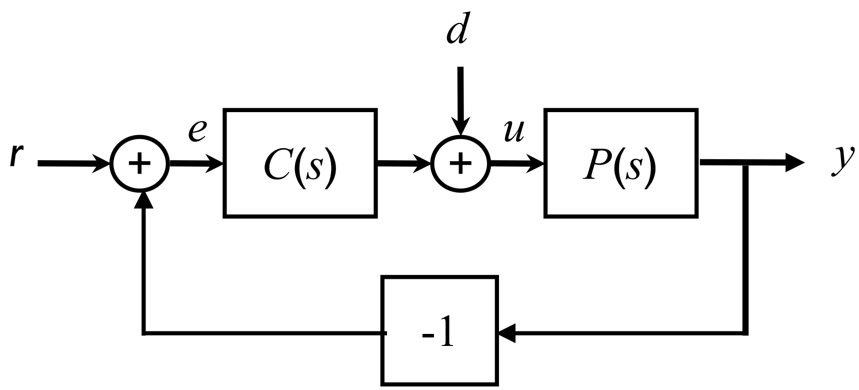
Then

$$Z = N + P$$



- Trace frequency response for $L(s)$ along the Nyquist "D" contour
- Count net # of clockwise encirclements of the -1 point

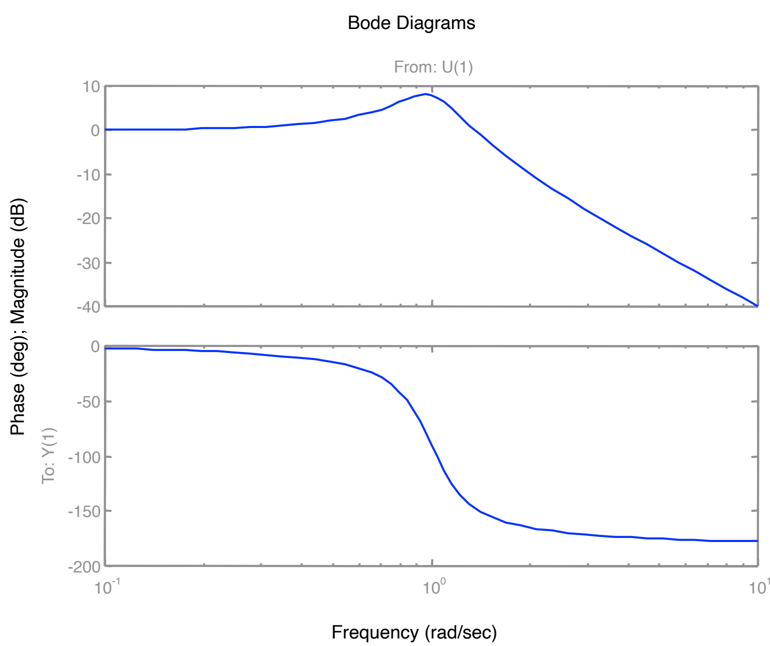
Simple Interpretation of Nyquist



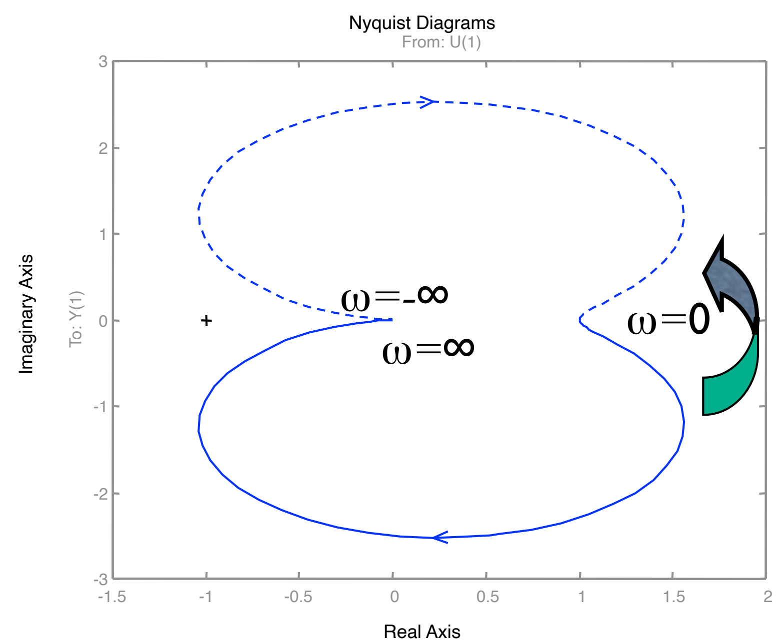
Basic idea: avoid positive feedback

- If L(s) has 180˚ phase (or greater) and gain greater than 1, then signals are amplified around loop
- Use when phase is monotonic
- General case requires Nyquist

Can generate Nyquist plot from Bode plot + reflection around real axis

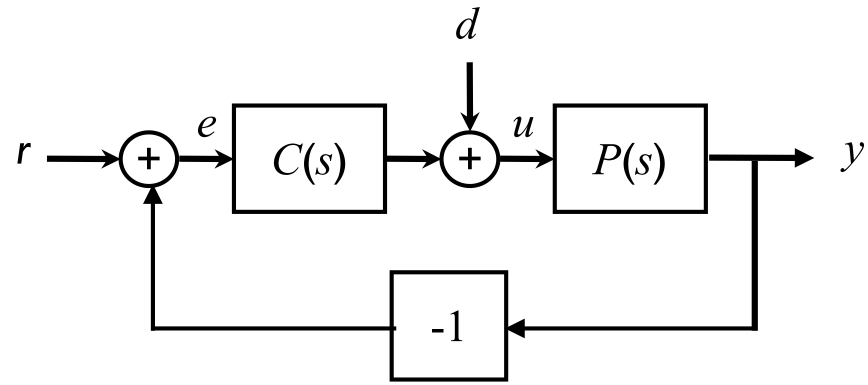


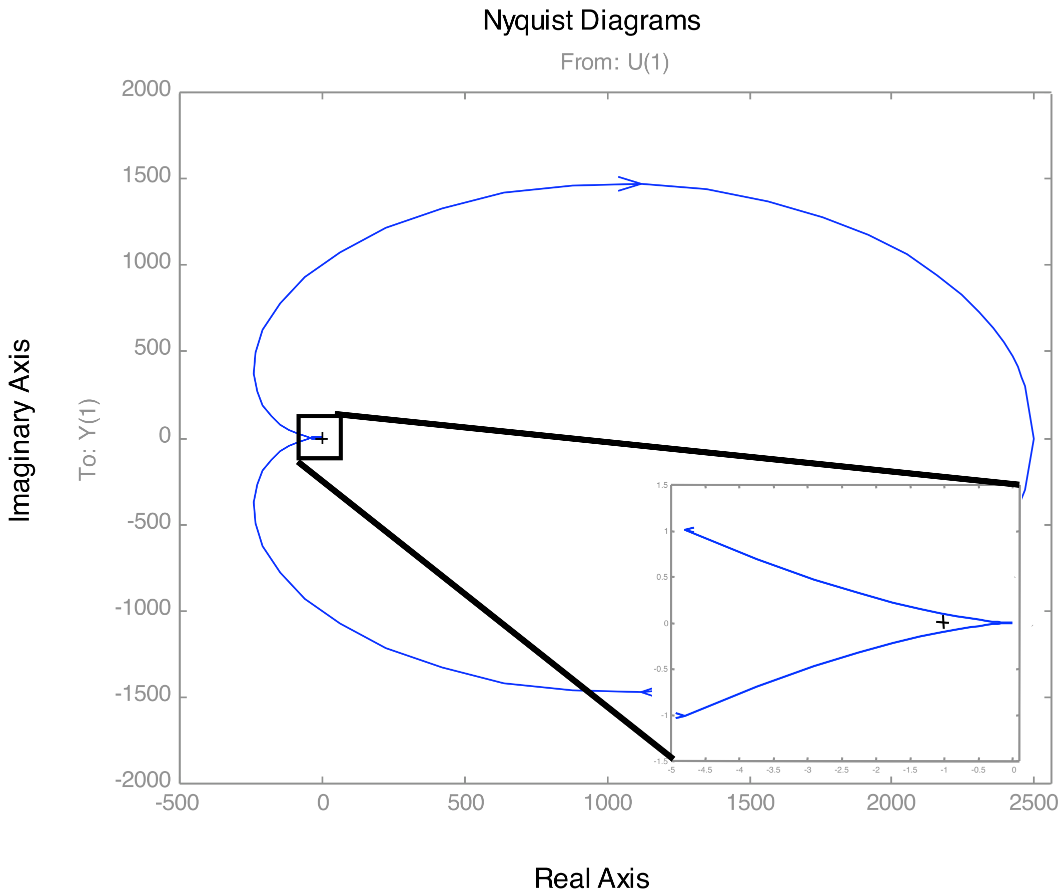
ambode(sys) [or bode(sys) in dB]



amnyquist(sys)

Example: Proportional + Integral* speed controller





$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s+a}$$

$$C(s) = K_p + \frac{K_i}{s + 0.01}$$

Remarks

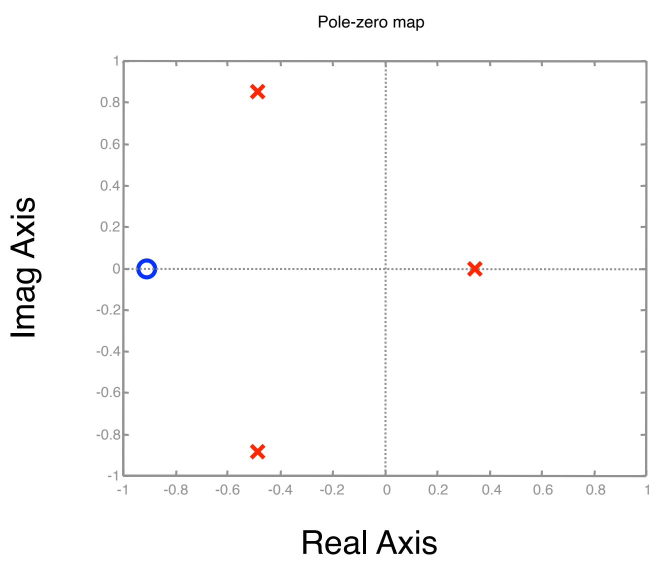
- N = 0, P = 0 ⇒ Z = 0 (stable)
- Need to zoom in to make sure there are no net encirclements
- Note that we don't have to compute closed loop response

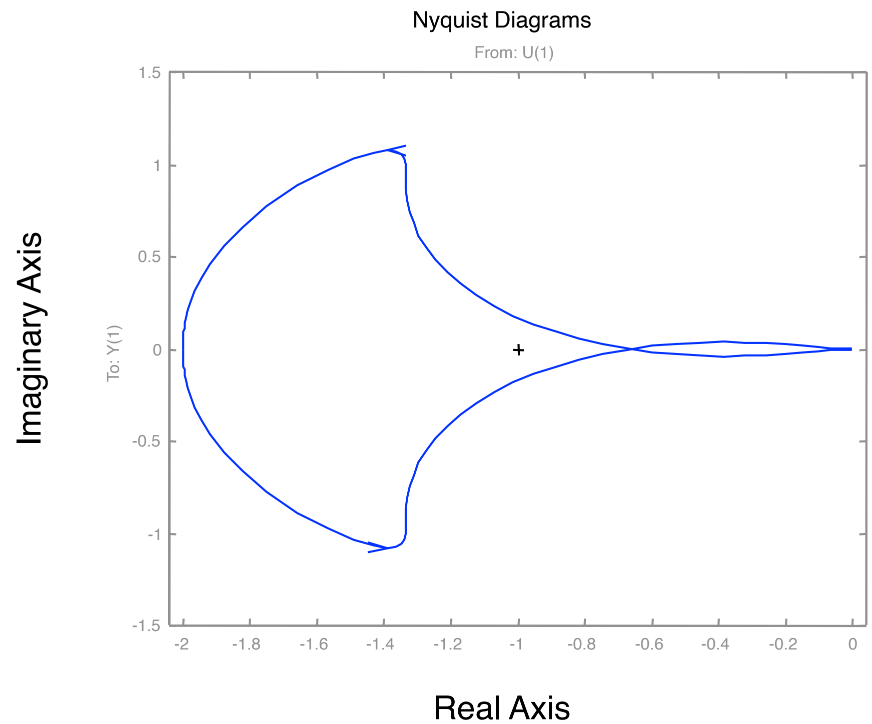
* slightly modified; more on the design of this compensator in next week's lecture

More complicated systems

What happens when open loop plant has RHP poles?

- 1 + PC has singularities inside D contour ⇒ these must be taken into account





$$L(s) = \frac{s+1}{s-0.5} \times \frac{1}{s^2+s+1}$$

unstable pole

N = -1, P = 1 ⇒ $Z = N+P$ = 0 (stable)

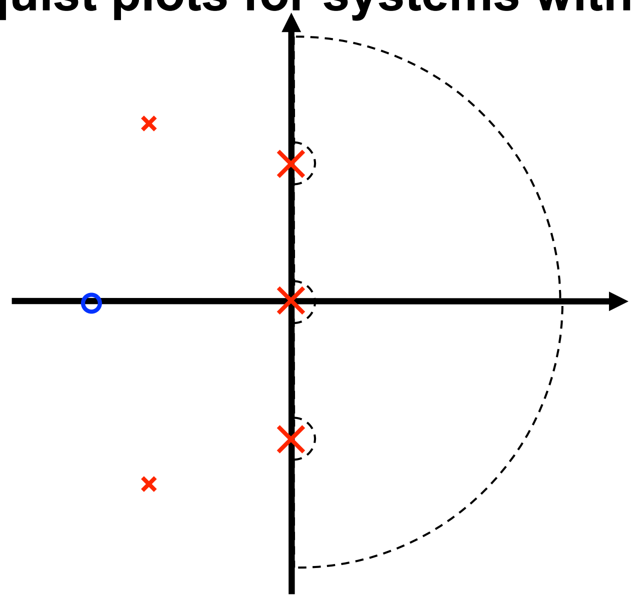
$$\frac{1}{1+L} = \frac{s+1}{(s+0.35)(s+0.07+1.2j)(s+0.07-1.2j)} ✓$$

Comments and cautions

Why is the Nyquist plot useful?

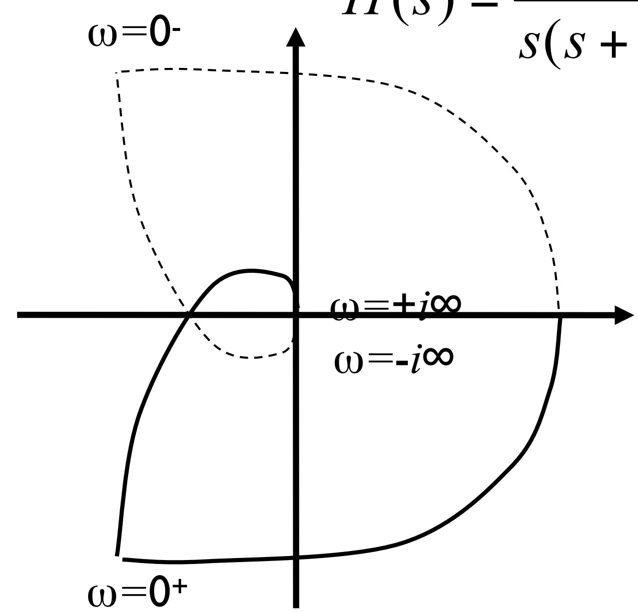
- Old answer: easy way to compute stability (before computers and MATLAB)
- Real answer: gives insight into stability and robustness; very useful for reasoning about stability

Nyquist plots for systems with poles on the jω axis



- chose contour to avoid poles on axis
- need to carefully compute Nyquist plot at these points
- evaluate $H(\varepsilon+0i)$ to determine direction

$$H(s) = \frac{1}{s(s+1)^2}$$



Cautions with using MATLAB

- MATLAB doesn't generate portion of plot for poles on imaginary axis
- These must be drawn in by hand (make sure to get the orientation right!)
- (Python-control toolbox does a much better job of plotting, but still need to take care)

Robust stability: gain and phase margins

Nyquist plot tells us if closed loop is stable, but not how stable

Gain margin

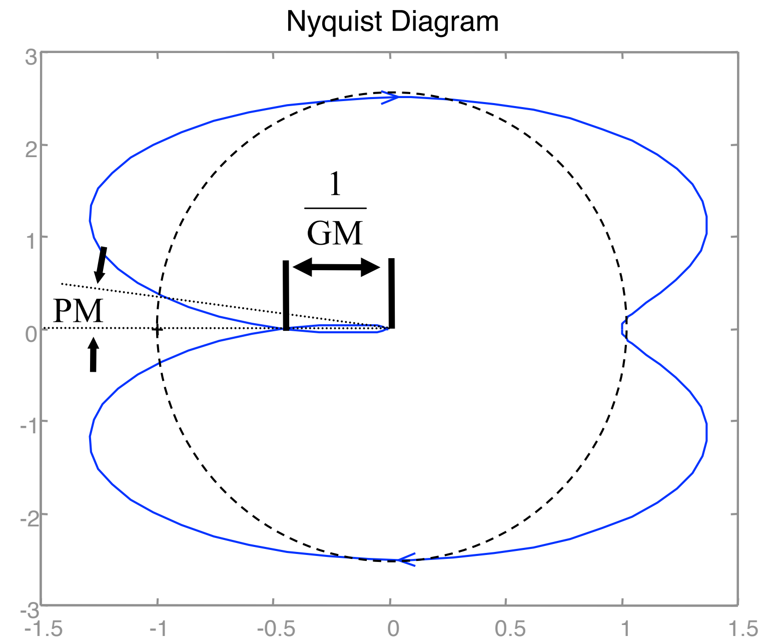
- How much we can modify the loop gain and still have the system be stable
- Determined by the location where the loop transfer function crosses 180˚ phase

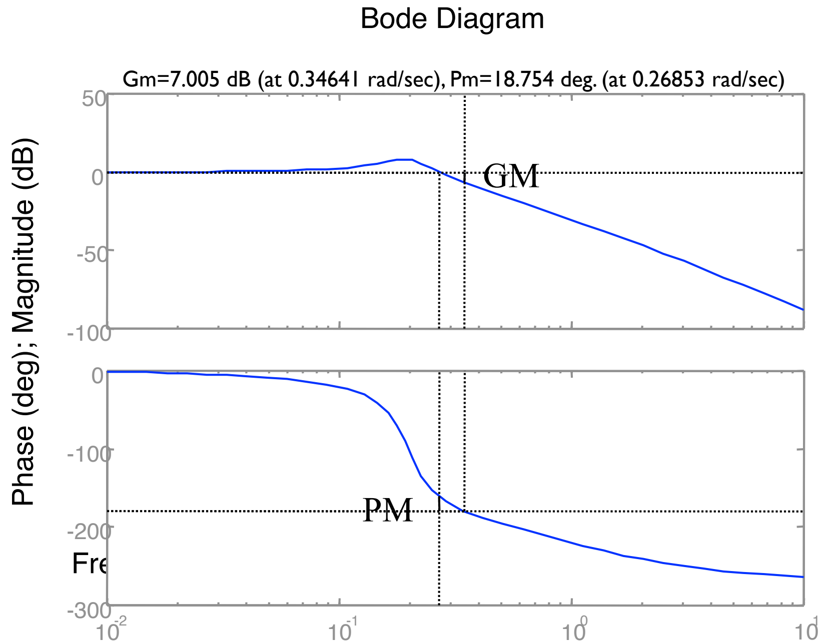
Phase margin

- How much we can add "phase delay" and still have the system be stable
- Determined by the phase at which the loop transfer function has unity gain

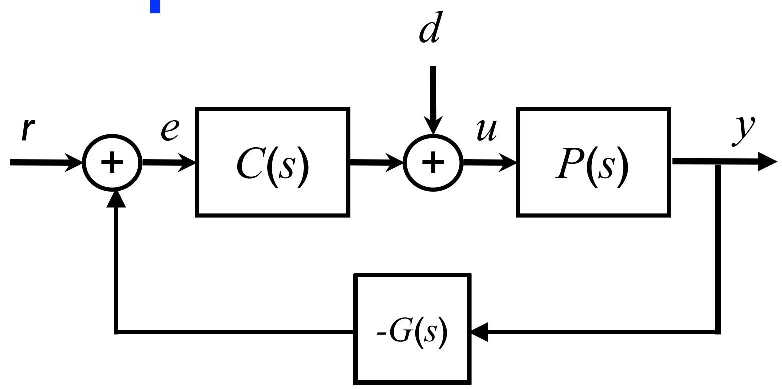
Bode plot interpretation

- Look for gain = 1, 180˚ phase crossings
- MATLAB: margin(sys)





Example: cruise control



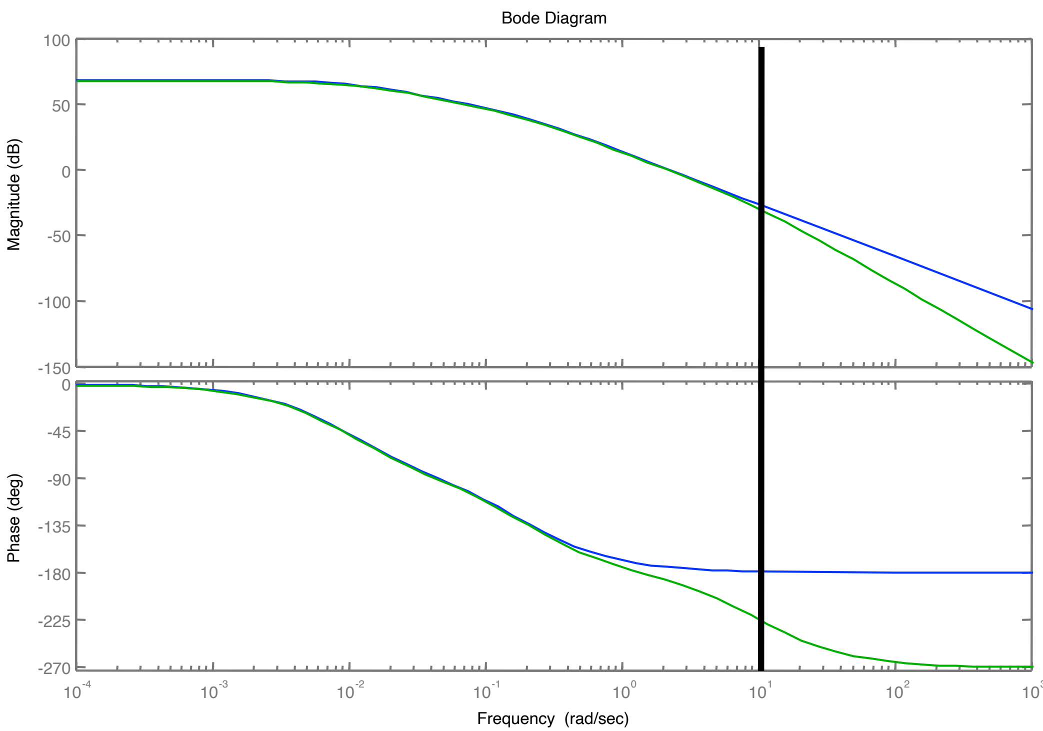
$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s + a}$$

$$C(s) = K_p + \frac{K_i}{s + 0.01}$$

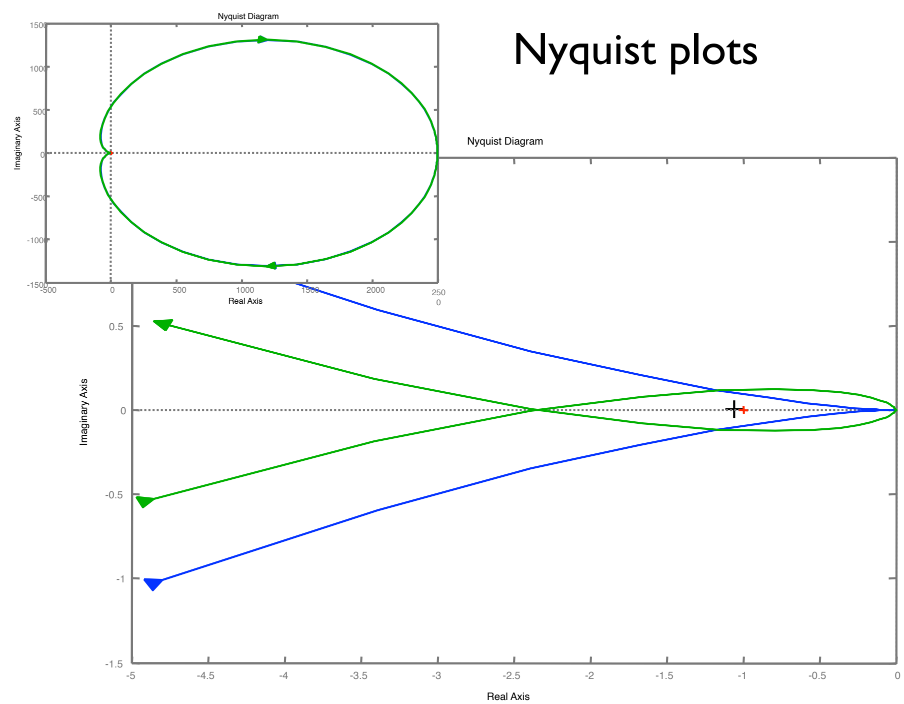
$$G(s) = \frac{10}{s + 10}$$

Effect of additional sensor dynamics

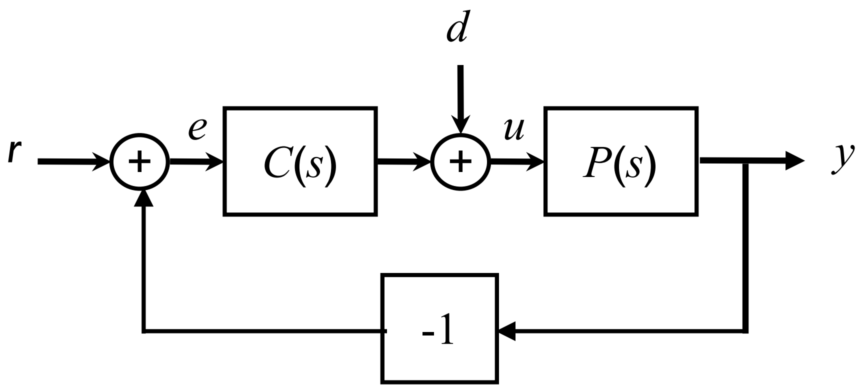
- New speedometer has pole at s = 10 (very fast); problems develop in the field
- What's the problem? A: insufficient phase margin in original design (not robust)

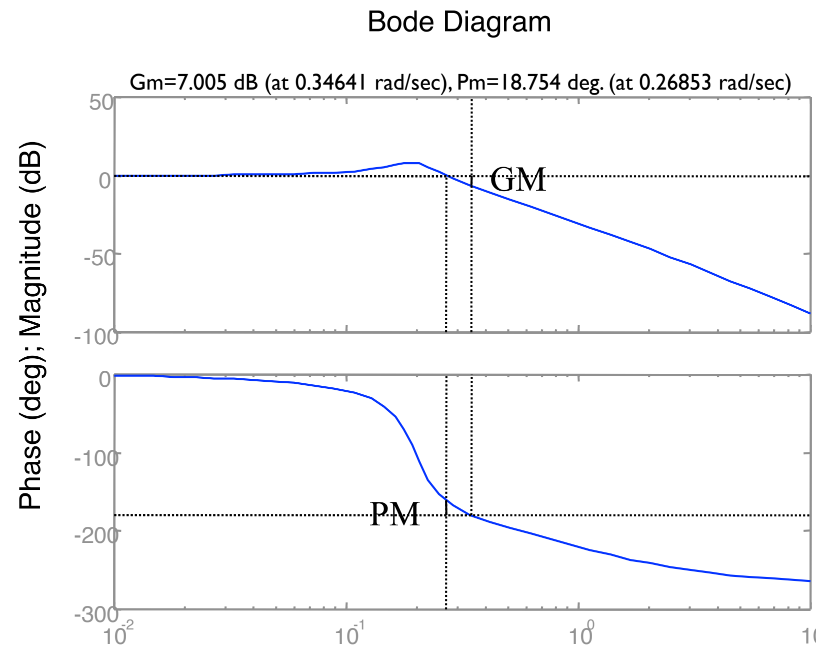


Nyquist plots

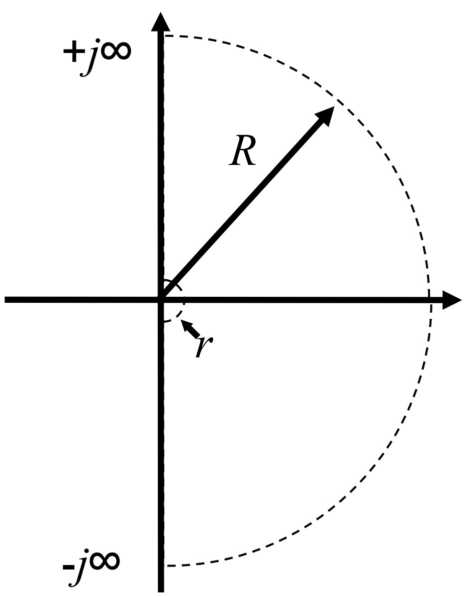


Summary: Loop Analysis of Feedback Systems





- Nyquist criteria for loop stability
- Gain, phase margin for robustness



Thm (Nyquist).

P # RHP poles of *L*(*s*)

N # CW encirclements

Z # RHP zeros

$$Z = N + P$$