CDS 110/ChE 105: Lecture 5-1
Observability and State Estimation

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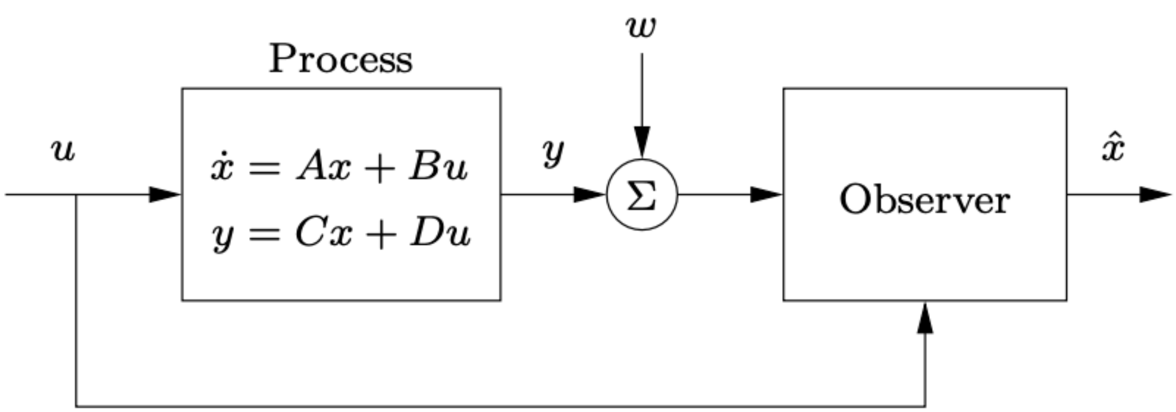
Goals:

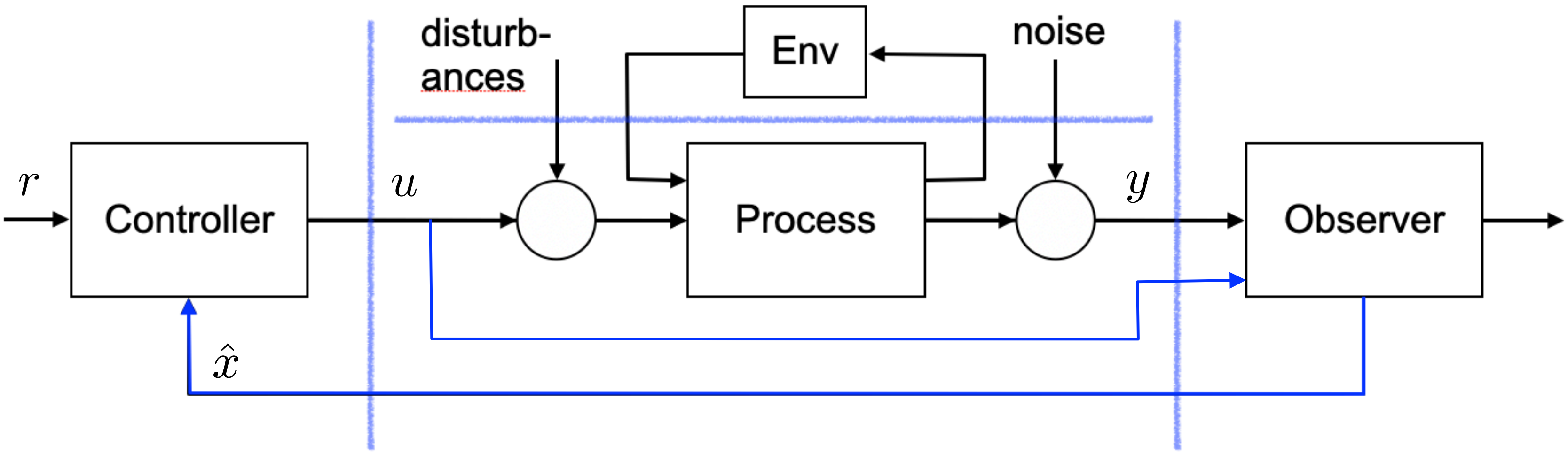
- Introduce the state estimation problem and observers
- Describe how to build observes using output feedback
- Describe how to design optimal observers (Kalman filter)

Reading:

- Åström and Murray, *Feedback Systems,* Sections 8.1-8.4



The State Estimation Problem



Problem Setup

- Given a dynamical system with noise and uncertainty, estimate the state

$$\dot{x} = Ax + Bu + Fv \qquad\longrightarrow\qquad \dot{\hat{x}} = \alpha(\hat{x}, y, u) \quad \longleftarrow \text{observer}$$

$$y = Cx + Du + Gw \qquad\qquad \lim_{t\to\infty} E(x - \hat{x}) = 0$$

expected value

- $\hat{x}$ is called the *estimate* of x

Remarks

- Several sources of uncertainty: noise, disturbances, process, initial condition
- Uncertainties are unknown, except through their effect on measured output
- First question: when is this even *possible?*

Observability

Defn A dynamical system of the form

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

is *observable* if for any $T > 0$ it is possible to determine the state of the system $x(T)$ through measurements of $y(t)$ and $u(t)$ on the interval $[0,T]$

Remarks

- Observability must respect *causality*: only get to look at past measurements
- We have ignored noise, disturbances for now ⇒ estimate exact state
- Intuitive way to check observability: assume we measure $y(t)$ and $u(t)$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad \begin{aligned} y &= \underline{Cx} \\ \dot{y} &= C\dot{x} = \underline{CAx} + CBu \\ \ddot{y} &= \underline{CA^2x} + CABu + CB\dot{u} \\ &\vdots \end{aligned} \qquad W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Thm A linear system is observable if and only if the observability matrix W_o is full rank. (For single-input, single-output, full rank => square and invertible)

Proof of Observability Rank Condition, 1/2

Thm A linear system is observable if and only if the observability matrix W_o is full rank.

Proof (sufficiency) Write the output in terms of the convolution integral

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

Since we know $u(t)$, we can subtract off its contribution and write

$$\tilde{y}(t) = Ce^{At}x(0)$$

Now differentiate the (new) output and evaluate at $t = 0$

$$\begin{aligned} \tilde{y}(0) &= Cx(0) \\ \dot{\tilde{y}}(0) &= CAx(0) \\ &\vdots \\ \tilde{y}^{(n)}(0) &= CA^{n-1}x(0) \end{aligned}$$

Finally, invert to solve for $x(0)$. To find $x(T)$, use $x(T) = e^{AT}x(0)$.

Proof of Observability Rank Condition, 2/2

Thm A linear system is observable if and only if the observability matrix W_o is full rank.

Proof (necessity) Again, we start with the convolution integral

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

Subtracting off the input as before and expanding the exponential, we have

$$\tilde{y}(t) = Ce^{At}x(0) = C(I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots)x(0)$$

By the Cayley-Hamilton theorem*, we can write A^n in terms of lower powers of A and so we can write

$$\tilde{y}(t) = (\alpha_0(t)C + \alpha_1(t)CA + \cdots + \alpha_{n-1}(t)CA^{n-1})x(0)$$

If W_o is not full rank, then can choose $x(0) \neq 0$ such that $\tilde{y}(0) = 0 \Rightarrow$ not observable (since $x(0) = 0$ would produce the same output).

* FBS2e, Exercise 7.3: $\lambda(A) = A^n + a_1A^{-1} + \cdots + a_{n-1}A^{n--1} + a_n = 0$

State Estimation: Full Order Observer

Given that a system is observable, how do we actually estimate the state?

- Key insight: if current estimate is correct, follow the dynamics of the system

$$\dot{x} = Ax + Bu \qquad y = Cx$$

$$\dot{\hat{x}} = \underbrace{A\hat{x} + Bu}_{} + L(y - C\hat{x})$$

correction (based on output error)

prediction (copy of dynamics)

- Modify the dynamics to correct for error based on a linear feedback term
- L is the *observer gain matrix*; determines how to adjust the state due to error
- Look at the error dynamics for $\tilde{x} = x - \hat{x}$ to determine how to choose L:

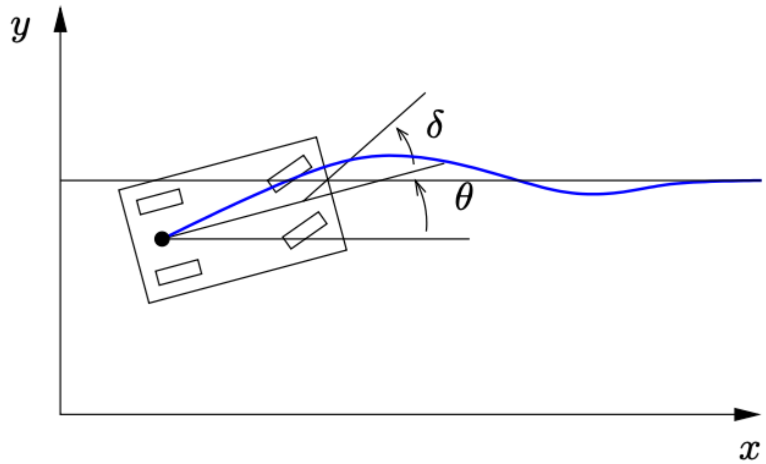
$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - (A\hat{x} + Bu + LC(x - \hat{x})) = (A - LC)\tilde{x}$$

Thm If the pair (A, C) is observable (associated W_o is full rank), then we can place the eigenvalues of A-LC arbitrarily through appropriate choice of L.

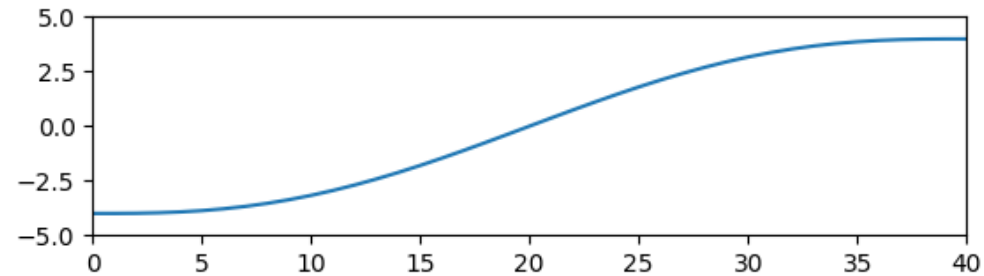
Proof Note that the transpose of A-LC is A^T - $C^T L^T$ and in this form, this is the same as the eigenvalue placement problem for state space controllers.

Remark: In Python, use `L = ct.place(A.T, C.T, eigs).T` to determine L

Example: Vehicle Lane Change Manuever



Lane change: -4 m to +4 m over 40 m



Equations of motion:

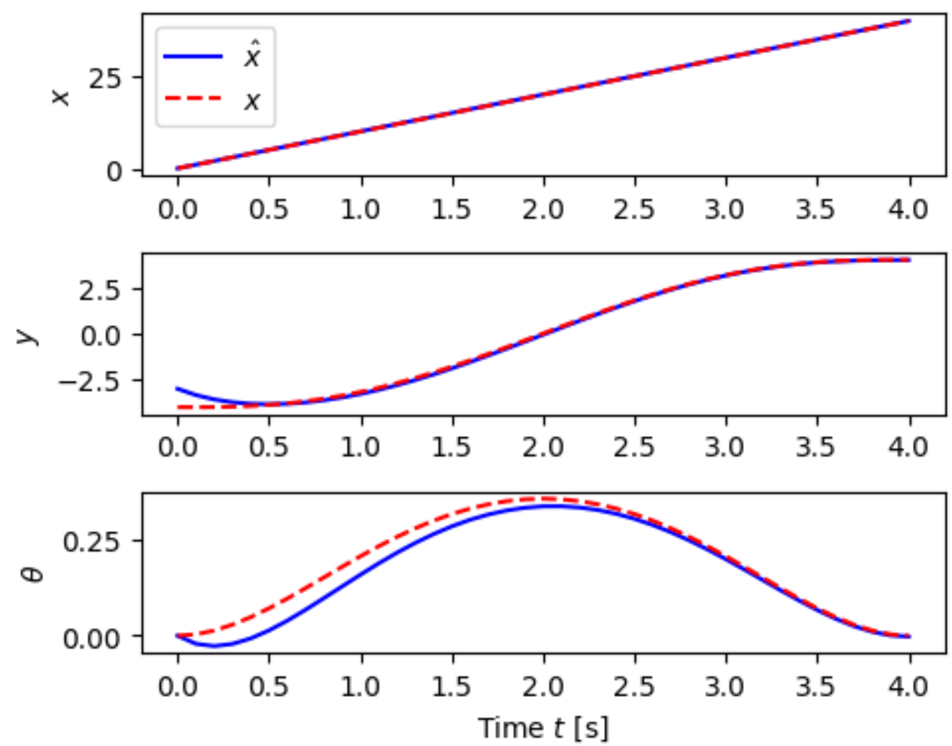
$$\dot{x} = \cos\theta\, v$$
$$\dot{y} = \sin\theta\, v$$
$$\dot{\theta} = \frac{v}{l}\tan\delta$$

Estimator design (L5-1_estimation.ipynb)

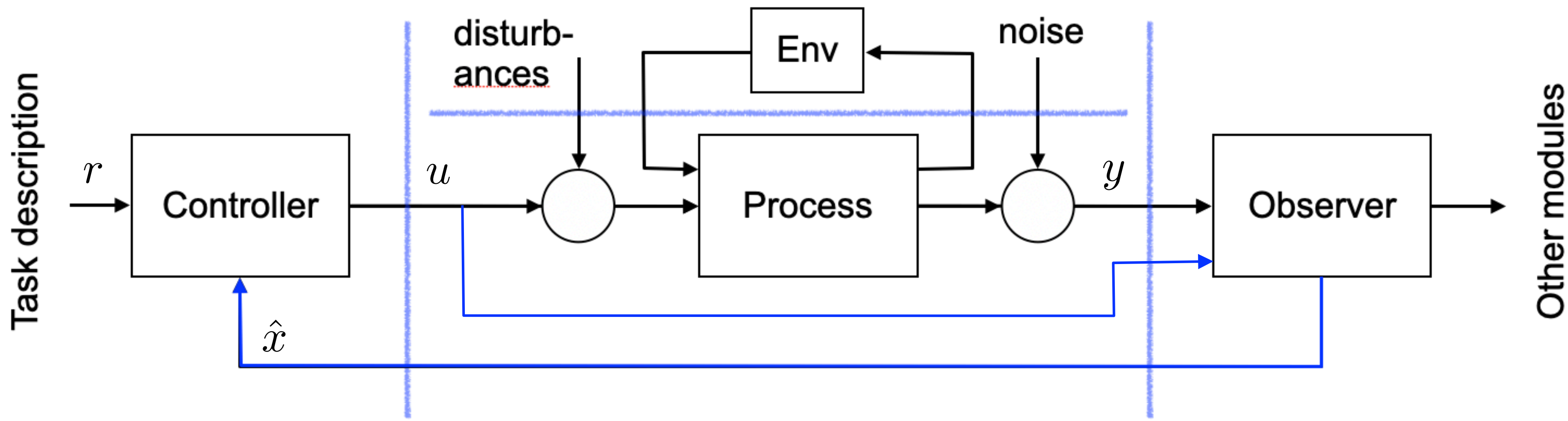
```
# Define the outputs to use for measurements
C = np.eye(2, 3)

# Compute the linearization of the nonlinear dynamics
P = kincar.linearize([0, 0, 0], [10, 0])

# Compute the gains via eigenvalue placement
L = ct.place(P.A.T, C.T, [-1, -2, -3]).T
```



The *Stochastic* State Estimation Problem



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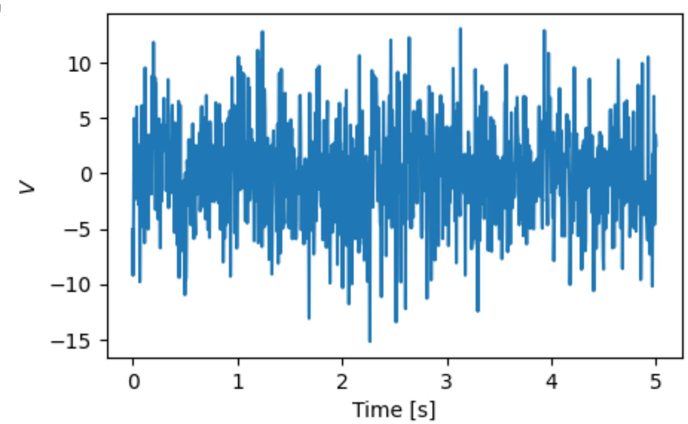
expected value

- $\hat{x}$ is called the *estimate* of x

Assume that disturbances and noise are "white noise'

- Signal has zero mean, but known covariance
- Covariance captures noisiness in the signal
- Full treatment in CDS 131; for how assume that noise parameters (covariance) is given



Optimal Estimation

System description

$$\dot{x} = Ax + Bu + Fv \qquad E\{v(s)v^T(t)\} = Q(t)\delta(t-s)$$
$$\dot{y} = Cx + w \qquad E\{w(s)w^T(t)\} = R(t)\delta(t-s)$$

- Disturbances and noise are multi-variable Gaussians with covariance Q, R

Problem statement: Find the estimate that minimizes the mean square error $E\{(x(t)-\widehat{x}(t))(x(t)-\widehat{x}(t))^T\}$ given $\{u(t), y(t): 0 \le t \le T\}$.

Proposition $\widehat{x}(t) = E\{x(t)|y(\tau), \tau \le t\}$

- Optimal estimate is just the expectation of the random process x given the constraint of the observed output.
- This is the way Kalman originally formulated the problem.
- Can think of this as a least squares problem: given all previous y(t), find the estimate $\widehat{x}(t)$ that satisfies the dynamics and minimizes the square error with the measured data.

Kalman-Bucy Filter

Theorem 1 (Kalman-Bucy, 1961). *The optimal estimator has the form of a linear observer*

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

where $L(t) = P(t)C^T R^{-1}$ & $P(t) = E\{(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T\}$ *satisfies*

$$\dot{P} = AP + PA^T - PC^T R^{-1}(t)CP + FQ(t)F^T$$
$$P(0) = E\{x(0)x^T(0)\}$$

Proof. (sketch) The error dynamics are given by

$$\dot{e} = (A - LC)e + \xi \qquad \xi = Fv - Lw \qquad R_\xi = FQF^T + LRL^T$$

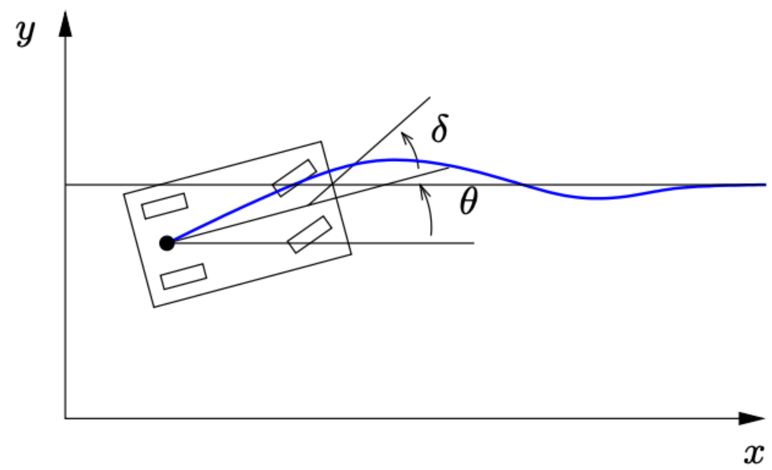
The covariance matrix $P_e = P$ for this process satisfies

$$\dot{P} = (A - LC)P + P(A - LC)^T + FQF^T + LRL^T.$$

We need to find L such that $P(t)$ is as small as possible. Can show that the L that acheives this is given by

$$RL^T = CP \qquad \Longrightarrow \qquad L = PC^T R^{-1}$$

Example: Vehicle Lane Change Manuever



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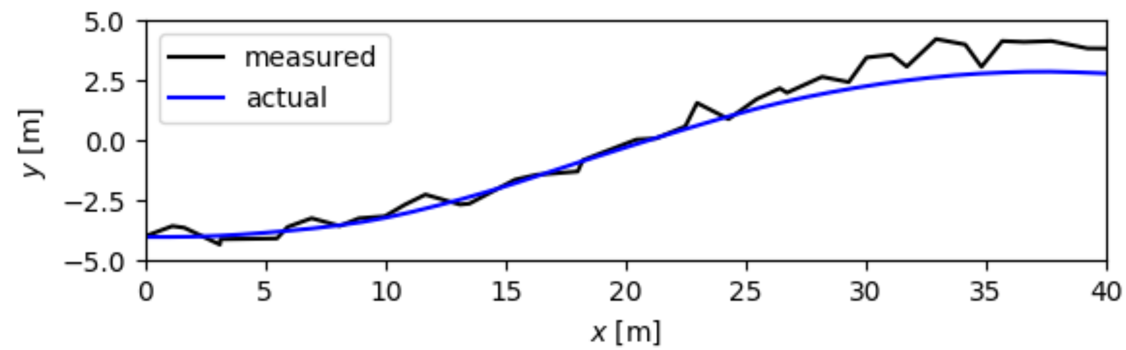

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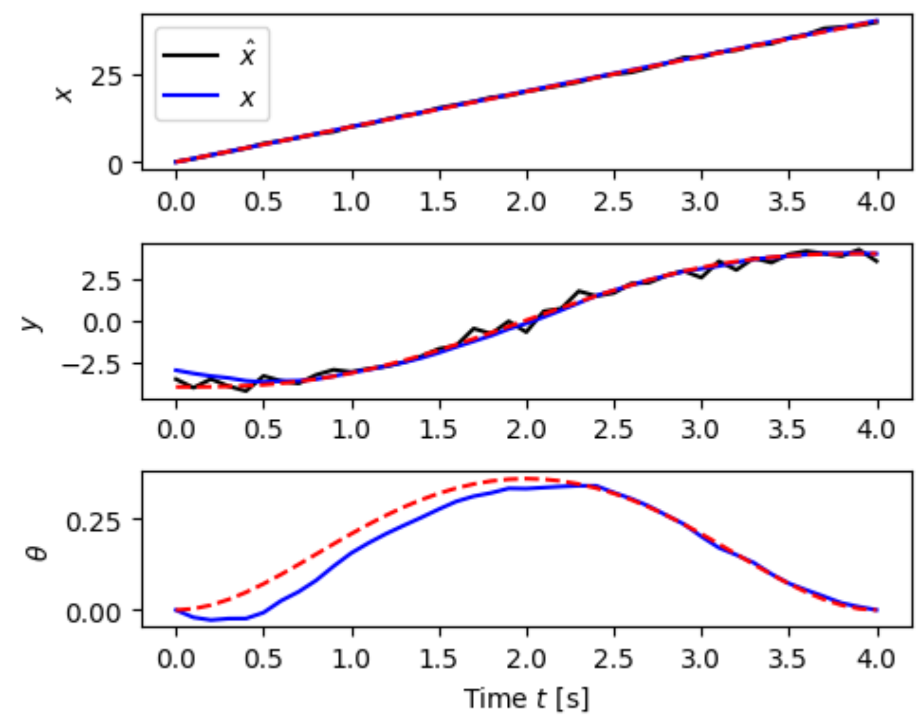

Estimator design (L5-1_estimation.ipynb)

```
# Disturbance and noise covariances
Qv = np.diag([0.1**2, 0.01**2])
Qw = np.eye(2) * 0.1**2

# Compute the Kalman gains
L_kf, _, _ = ct.lqe(P.A, P.B, C, Qv, Qw)
```

Summary: Observers and State Estimation

Estimation of the state based on model + input/output measurements

- If system is observable, can estimate state via $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$
- Choose gains via eigenvalue placement or optimal estimation (Kalman filter)

Wednesday: combine with state feeback

- Controller consists of estimator + state feedback using estimated state
- Can show that if estimator is stable and controller is stabilizing, closed loop system is stable
- Many tradeoffs to explore in design of controller

Fri: walk through kinematic car example