Part 1: LQR Tracking Example

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This example uses a linear system to show how to implement LQR based tracking and some of the tradeoffs between feedfoward and feedback. Integral action is also implemented.

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt

try:
  import control as ct
  print("python-control version:", ct.__version__)
except ImportError:
  # Version 0.10.0 is enough for this notebook
  !pip install control
  import control as ct
```

```
python-control version: 0.10.0
```

System definition

We'll use a simple linear system to illustrate the concepts:

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 10 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x.$$

In [2]:
```
# Define a simple linear system that we want to control
A = np.array([[0, 10], [-1, 0]])
B = np.array([[0], [1]])
C = np.array([[1, 1]])
sys = ct.ss(A, B, C, 0, name='sys')
print(sys)
```

```
<StateSpace>: sys
Inputs (1): ['u[0]']
Outputs (1): ['y[0]']
States (2): ['x[0]', 'x[1]']

A = [[ 0. 10.]
     [-1.  0.]]

B = [[0.]
     [1.]]

C = [[1. 1.]]

D = [[0.]]
```

Linear quadratic regulator (LQR) design

We'll design a controller of the form

$$u = -Kx + k_r r$$

- For the feedback control gain K, we'll use linear quadratic regulator theory. We seek to find the control law that minimizes the cost function:

  $$J(x(\cdot), u(\cdot)) = \int_0^\infty x^T(\tau) Q x(\tau) + u^T(\tau) R u(\tau)\, d\tau$$

  The weighting matrices $Q \succeq 0 \in \mathbb{R}^{n \times n}$ and $R \succ 0 \in \mathbb{R}^{m \times m}$ should be chosen based on the desired performance of the system (tradeoffs in state errors and input magnitudes). See Example 3.5 in OBC for a discussion of how to choose these weights. For now, we just choose identity weights for all states and inputs.

- For the feedforward control gain k_r, we'll use the technique discussed in class on wednesday, where the feedforward gain is derived from an equilibrium point analysis:

$$\begin{aligned} y_e &= C(A - BK)^{-1} B k_r r \\ \Rightarrow k_r &= \frac{-1}{C(A - BK)^{-1} B} \end{aligned}$$

In [3]:
```
# Construct an LQR controller for the system
Q = np.eye(sys.nstates)
R = np.eye(sys.ninputs)
K, _, _ = ct.lqr(sys, Q, R)
print('K: '+str(K))

# Set the feedforward gain to track the reference
kr = (-1 / (C @ np.linalg.inv(A - B @ K) @ B))
print('k_r: '+str(kr))
```

```
K: [[0.41421356 3.04701021]]
k_r: [[1.41421356]]
```

Now that we have our gains designed, we can simulate the closed loop system:

$$\frac{dx}{dt} = A_{cl}x + B_{cl}r, \quad A_{cl} = A - BK, \quad B_{cl} = Bk_r$$

Notice that, with a state feedback controller, the new (closed loop) dynamics matrix absorbs the old (open loop) "input" u, and the new (closed loop) input is our reference signal r.

In [4]:
```
# Create a closed loop system
A_cl = A - B @ K
B_cl =  B * kr
clsys = ct.ss(A_cl, B_cl, C, 0)
print(clsys)
```

```
<StateSpace>: sys[0]
Inputs (1): ['u[0]']
Outputs (1): ['y[0]']
States (2): ['x[0]', 'x[1]']

A = [[ 0.          10.        ]
     [-1.41421356 -3.04701021]]

B = [[0.        ]
     [1.41421356]]

C = [[1. 1.]]

D = [[0.]]
```

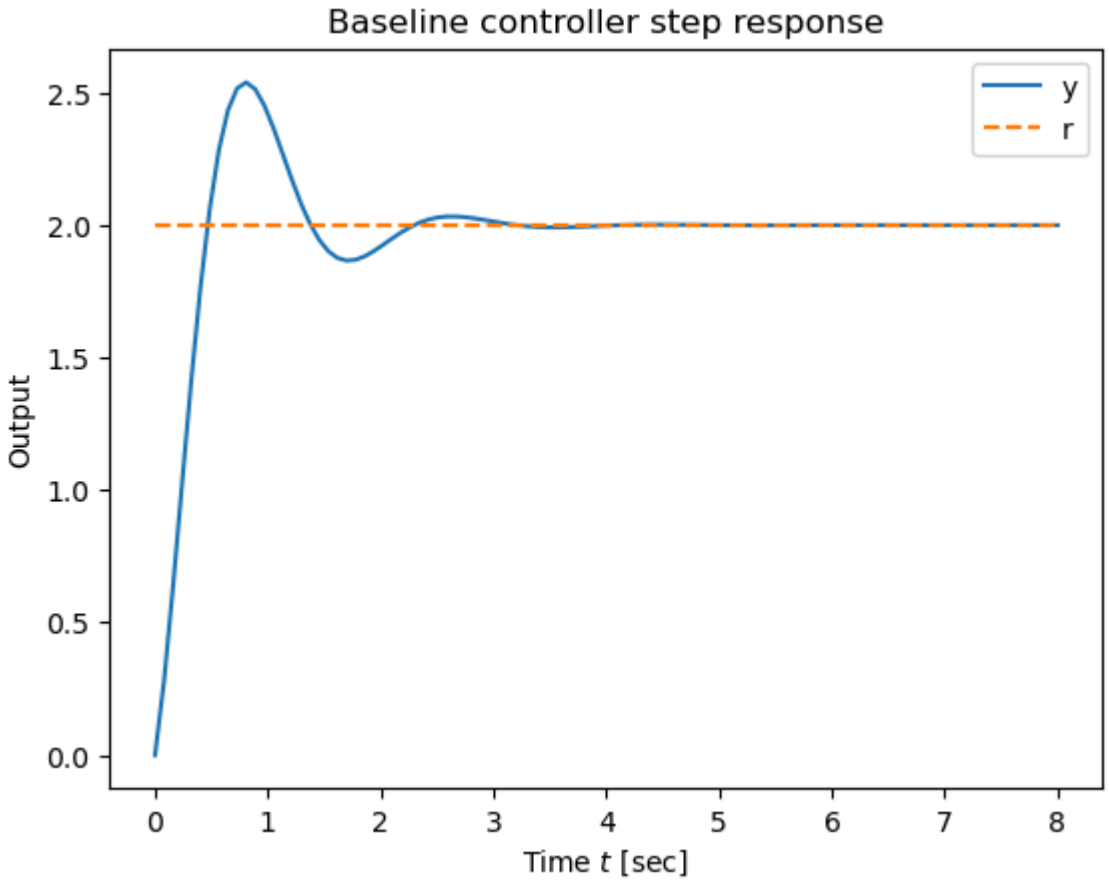
System simulations

Baseline controller

To see how the baseline controller performs, we ask it to track a constant reference:

In [5]:
```
# Plot the step response with respect to the reference input
r = 2
Tf = 8
tvec = np.linspace(0, Tf, 100)

U = r * np.ones_like(tvec)
time, output = ct.input_output_response(clsys, tvec, U)
plt.plot(time, output)
plt.plot([time[0], time[-1]], [r, r], '--');
plt.legend(['y', 'r']);
plt.ylabel("Output")
plt.xlabel("Time $t$ [sec]")
plt.title("Baseline controller step response")
```

Out[5]: Text(0.5, 1.0, 'Baseline controller step response')



(If there's time:)

- try setting $k_r = 0$
- try setting $k_r \neq \frac{-1}{C(A-BK)^{-1}B}$
- try different LQR weightings

Disturbance rejection

To add an input disturbance to the system, we include a second open loop input:

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 10 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix}, \qquad y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x.$$

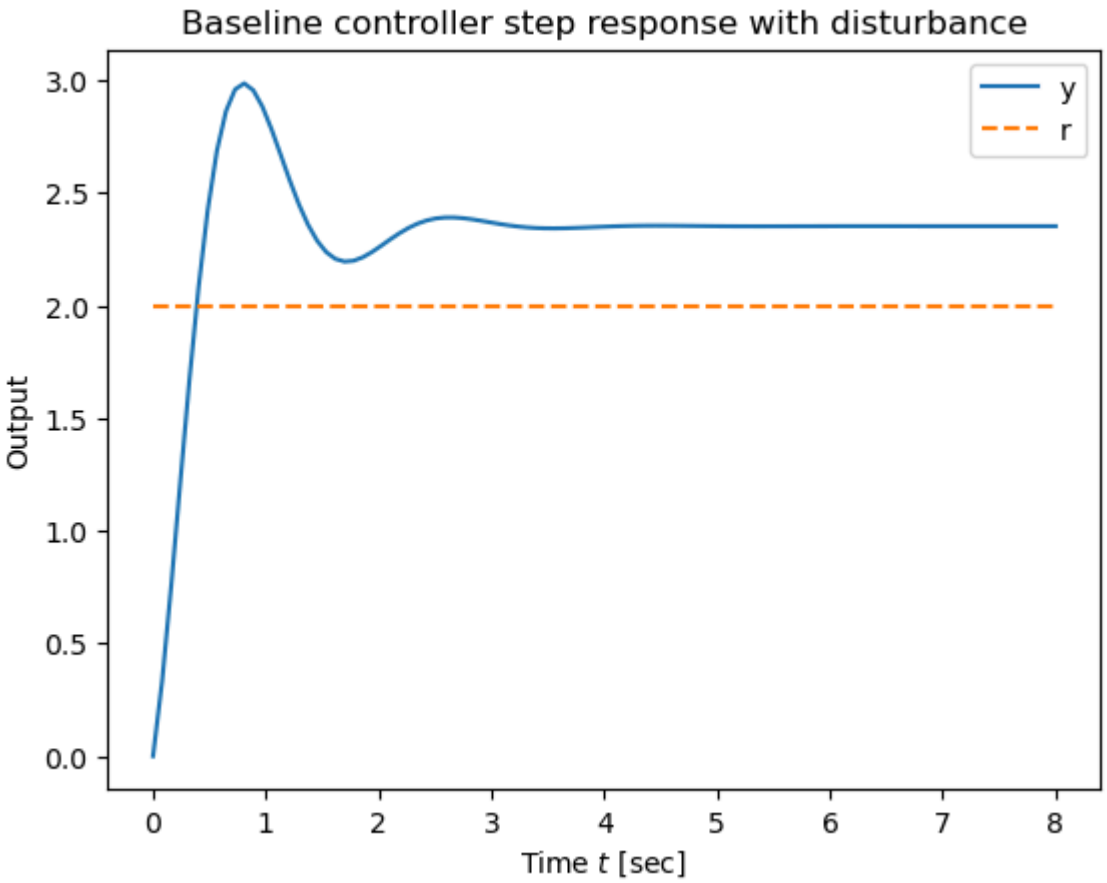
Our closed loop system becomes:

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 10 \\ -1 - K_1 & 0 - K_2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ k_r & 1 \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}, \qquad y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x.$$

In [6]:
```
# Resimulate with a disturbance input
B_ext = np.hstack([B * kr, B])
clsys = ct.ss(A - B @ K, B_ext, C, 0)
```

```
# Construct the inputs for the augmented system
delta = 0.5
U = np.vstack([r * np.ones_like(tvec), delta * np.ones_like(tvec)])

time, output = ct.input_output_response(clsys, tvec, U)

plt.plot(time, output[0])
plt.plot([time[0], time[-1]], [r, r], '--')
plt.legend(['y', 'r']);
plt.ylabel("Output")
plt.xlabel("Time $t$ [sec]")
plt.title("Baseline controller step response with disturbance")
```

Out[6]: `Text(0.5, 1.0, 'Baseline controller step response with disturbance')`



We see that this leads to steady state error, since the feedforward signal didn't include an offset for the disturbance.

Integral feedback

A standard approach to compensate for constant disturbances is to use integral feedback. To do this, we have to keep track of the integral of the error

$$z = \int_0^\tau (y - r)\, d\tau = \int_0^\tau (Cx - r)\, d\tau.$$

We do this by creating an augmented system that includes the dynamics of the process (dx/dt) along with the dynamics of the integrator state (dz/dt):

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -I \end{bmatrix} r, \qquad y = \begin{bmatrix} C \\ 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}.$$

In [7]:
```
# Define an augmented state space for use with LQR
A_aug = np.block([[sys.A, np.zeros((sys.nstates, 1))], [C, 0] ])
B_aug = np.vstack([sys.B, 0])
print("A =", A_aug, "\nB =", B_aug)
```

```
A = [[ 0. 10.  0.]
 [-1.  0.  0.]
 [ 1.  1.  0.]]
B = [[0.]
 [1.]
 [0.]]
```

Our controller then takes the form:

$$\begin{aligned} u &= -Kx - k_\mathrm{i}\int_0^\tau (y - r)\, d\tau + k_r r \\ &= -(Kx + k_\mathrm{i} z) + k_r r. \end{aligned}$$

This results in the closed loop system:

$$\frac{dx}{dt} = \begin{bmatrix} A - BK & -Bk_i \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} Bk_r \\ -I \end{bmatrix} r, \qquad y = \begin{bmatrix} C \\ 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}.$$

Since z is part of the augmented state space, we can generate an LQR controller for the the augmented system to find both the usual gain K and the integral gain k_i:

$$\bar{K} = \begin{bmatrix} K & k_i \end{bmatrix}$$

In [8]:
```
# Create an LQR controller for the augmented system
K_aug, _, _ = ct.lqr(A_aug, B_aug, np.diag([1, 1, 1]), np.eye(sys.ninputs))
print('K_aug: '+str(K_aug))
K = K_aug[:, 0:2]
ki = K_aug[:, 2]
kr = -1 / (C @ np.linalg.inv(A - B * K) @ B)
```

```
K_aug: [[0.56288531 3.77593779 1.        ]]
```

Notice that the value of K changed, so we needed to recompute k_r too.

To run simulations, we return to our system augmented with a disturbance, but we expand the outputs available to the controller:

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 10 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ d \end{bmatrix},$$

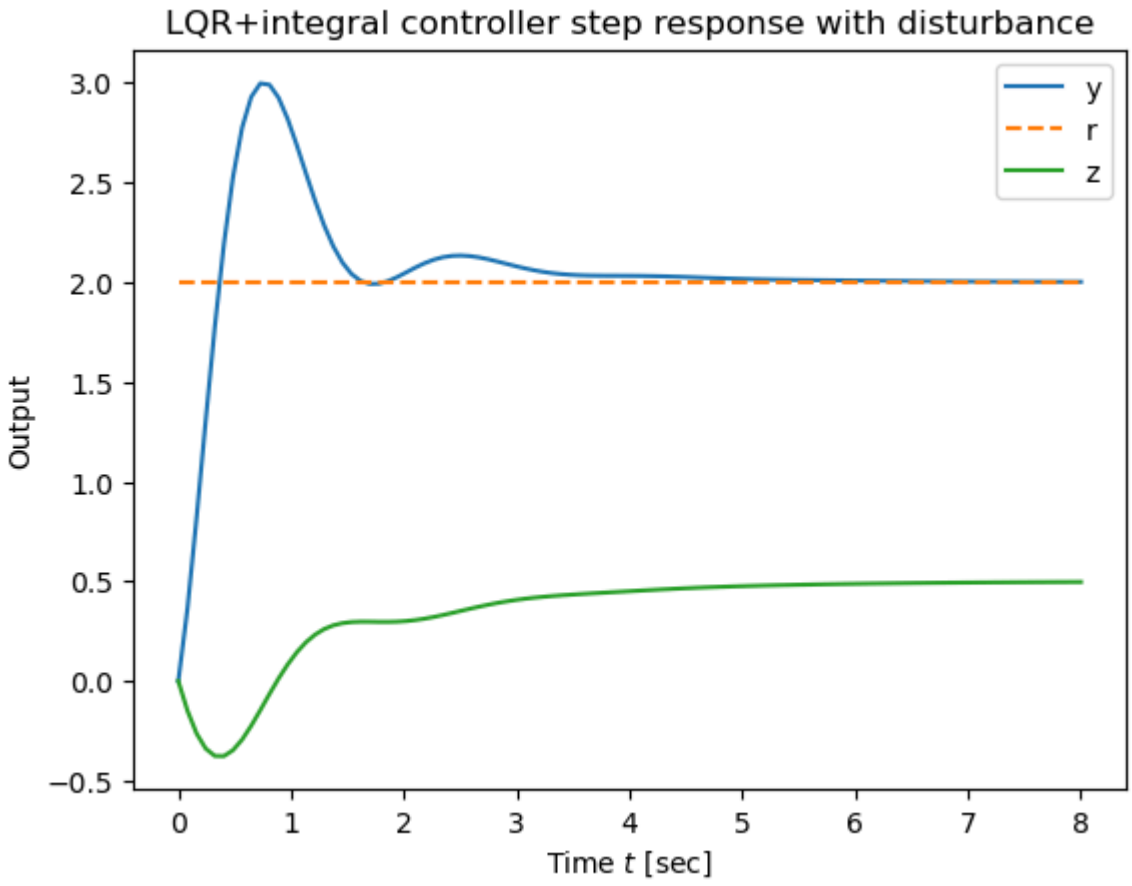
$$\bar{y} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^T x = \begin{bmatrix} x_1 & x_2 & y \end{bmatrix}.$$

The controller then constructs its internal state z out of x and r.

In [9]:
```
# Construct a system with disturbance inputs, and full outputs (for the cont
A_integral = sys.A
B_integral = np.hstack([sys.B, sys.B])
C_integral = [[1, 0], [0, 1], [1, 1]] # outputs for the controller: x1, x2,
sys_integral = ct.ss(
    A_integral, B_integral, C_integral, 0,
    inputs=['u', 'd'],
    outputs=['x1', 'x2', 'y']
)
print(sys_integral)

# Construct an LQR+integral controller for the system with an internal state
A_ctrl = [[0]]
B_ctrl = [[1, 1, -1]] # z_dot=Cx-r
C_ctrl = -ki #-ki*z
D_ctrl = np.hstack([-K, kr]) #-K*x + kr*r
ctrl_integral=ct.ss(
    A_ctrl, B_ctrl, C_ctrl, D_ctrl, # u = -ki*z - K*x + kr*r
    inputs=['x1', 'x2', 'r'],    # system outputs + reference
    outputs=['u'],               # controller action
)
print(ctrl_integral)

# Create the closed loop system
clsys_integral = ct.interconnect([sys_integral, ctrl_integral], inputs=['r',
print(clsys_integral)

# Resimulate with a disturbance input
delta = 0.5
U = np.vstack([r * np.ones_like(tvec), delta * np.ones_like(tvec)])
time, output, states = ct.input_output_response(clsys_integral, tvec, U, ret
plt.plot(time, output[0])
plt.plot([time[0], time[-1]], [r, r], '--')
plt.plot(time, states[2])
plt.legend(['y', 'r', 'z']);
plt.ylabel("Output")
plt.xlabel("Time $t$ [sec]")
plt.title("LQR+integral controller step response with disturbance")
```

```
<StateSpace>: sys[2]
Inputs (2): ['u', 'd']
Outputs (3): ['x1', 'x2', 'y']
States (2): ['x[0]', 'x[1]']

A = [[ 0. 10.]
     [-1.  0.]]

B = [[0. 0.]
     [1. 1.]]

C = [[1. 0.]
     [0. 1.]
     [1. 1.]]

D = [[0. 0.]
     [0. 0.]
     [0. 0.]]

<StateSpace>: sys[3]
Inputs (3): ['x1', 'x2', 'r']
Outputs (1): ['u']
States (1): ['x[0]']

A = [[0.]]

B = [[ 1.  1. -1.]]

C = [[-1.]]

D = [[-0.56288531 -3.77593779  1.56288531]]

<LinearICSystem>: sys[4]
Inputs (2): ['r', 'd']
Outputs (1): ['y']
States (3): ['sys[2]_x[0]', 'sys[2]_x[1]', 'sys[3]_x[0]']

A = [[ 0.         10.          0.        ]
     [-1.56288531 -3.77593779 -1.        ]
     [ 1.          1.          0.        ]]

B = [[ 0.          0.        ]
     [ 1.56288531  1.        ]
     [-1.          0.        ]]

C = [[1. 1. 0.]]

D = [[0. 0.]]
```

Out[9]: Text(0.5, 1.0, 'LQR+integral controller step response with disturbance')



Notice that the steady state value of $z = \int(y - r)$ is not zero, but rather settles to whatever value makes $y - r$ zero!

Part 2: PVTOL Linear Quadratic Regulator Example

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This notebook contains an example of LQR control applied to the PVTOL system. It demonstrates how to construct an LQR controller by linearizing the system, and provides an alternate view of the feedforward component of the controller.

System description

We use the PVTOL dynamics from the textbook, which are contained in the `pvtol` module.

$$
\begin{aligned}
m\ddot{x} &= F_1 \cos\theta - F_2 \sin\theta - c\dot{x}, \\
m\ddot{y} &= F_1 \sin\theta + F_2 \cos\theta - mg - c\dot{y}, \\
J\ddot{\theta} &= rF_1.
\end{aligned}
\qquad
\frac{dz}{dt} = \begin{bmatrix} z_4 \\ z_5 \\ z_6 \\ -\frac{c}{m} z_4 \\ -g - \frac{c}{m} z_5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{F_1}{m} cos\theta - \frac{F_2}{m} sin\theta \\ \frac{F_1}{m} sin\theta + \frac{F_2}{m} cos\theta \\ -\frac{r}{J} F_1 \end{bmatrix}
$$

The state space variables for this system are:

$$z = (x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}), \quad u = (F_1, F_2)$$

Notice that the x and y positions (z_1 and z_2) do not actually appear in the dynamics--this makes sense, since the aircraft should hypothetically fly the same way no matter where in the air it is (neglecting effects near the ground).

In [10]:
```
!wget --no-check-certificate https://www.cds.caltech.edu/~murray/courses/cds
from pvtol import pvtol, plot_results
print(pvtol)
print(pvtol.params)
```

```
--2024-05-04 16:00:24--  https://www.cds.caltech.edu/~murray/courses/cds112/
wi2023/pvtol.py
Resolving www.cds.caltech.edu (www.cds.caltech.edu)... 131.215.243.5
Connecting to www.cds.caltech.edu (www.cds.caltech.edu)|131.215.243.5|:44
3...
connected.
WARNING: The certificate of ‘www.cds.caltech.edu’ is not trusted.
WARNING: The certificate of ‘www.cds.caltech.edu’ doesn't have a known issue
r.
HTTP request sent, awaiting response... 200 OK
Length: 10910 (11K) [text/plain]
Saving to: ‘pvtol.py.1’


pvtol.py.1            0%[                    ]       0  --.-KB/s
pvtol.py.1          100%[===================>]  10.65K  --.-KB/s    in 0s

2024-05-04 16:00:24 (107 MB/s) - ‘pvtol.py.1’ saved [10910/10910]

<FlatSystem>: pvtol
Inputs (2): ['F1', 'F2']
Outputs (6): ['x0', 'x1', 'x2', 'x3', 'x4', 'x5']
States (6): ['x0', 'x1', 'x2', 'x3', 'x4', 'x5']

Update: <function _pvtol_update at 0x11e77fa60>
Output: <function _pvtol_output at 0x11e77fb00>

Forward: <function _pvtol_flat_forward at 0x11e77fba0>
Reverse: <function _pvtol_flat_reverse at 0x11e7c91c0>
{'m': 4.0, 'J': 0.0475, 'r': 0.25, 'g': 9.8, 'c': 0.05}
```

Since we will be creating a linear controller, we need a linear system model. We obtain that model by linearizing the dynamics around an equilibrium point.

The equilibrium point is the "hover" state, where F_2 (the vertical force) balances gravity:

$$z_e = (x_e, y_e, 0, 0, 0, 0), \quad u_e = (0, mg)$$

This can be done in python-control using the `find_eqpt` function. We fix the output of the system to be zero and find the state and inputs that hold us there.

In [11]:
```
# Find the equilibrium point corresponding to hover
xeq, ueq = ct.find_eqpt(pvtol, np.zeros(6), np.zeros(2), y0=np.zeros(6), iy=

print("xeq = ", xeq)
print("ueq = ", ueq)
```

```
xeq =  [0. 0. 0. 0. 0. 0.]
ueq =  [ 0.  39.2]
```

Next, we'll linearize the system around the equilibrium points. As discussed in FBS2e (example 7.9), the linearization around this equilibrium point has the form:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g & -c/m & 0 & 0 \\ 0 & 0 & 0 & 0 & -c/m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/m \\ r/J & 0 \end{bmatrix}.$$

(note that here r is a system parameter, not the same as the reference r we've been using elsewhere in this notebook)

In [12]:
```
# Get the linearized dynamics
linsys = pvtol.linearize(xeq, ueq)
print(linsys)
```

```
<StateSpace>: sys[6]
Inputs (2): ['u[0]', 'u[1]']
Outputs (6): ['y[0]', 'y[1]', 'y[2]', 'y[3]', 'y[4]', 'y[5]']
States (6): ['x[0]', 'x[1]', 'x[2]', 'x[3]', 'x[4]', 'x[5]']

A = [[ 0.00000000e+00  0.00000000e+00  0.00000000e+00  1.00000000e+00
       0.00000000e+00  0.00000000e+00]
     [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
       1.00000000e+00  0.00000000e+00]
     [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
       0.00000000e+00  1.00000000e+00]
     [ 0.00000000e+00  0.00000000e+00 -9.80000000e+00 -1.25000000e-02
       0.00000000e+00  0.00000000e+00]
     [ 0.00000000e+00  0.00000000e+00 -4.90096852e-06  0.00000000e+00
      -1.25000000e-02  0.00000000e+00]
     [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
       0.00000000e+00  0.00000000e+00]]

B = [[0.         0.        ]
     [0.         0.        ]
     [0.         0.        ]
     [0.25       0.        ]
     [0.         0.25      ]
     [5.26315789 0.        ]]

C = [[1. 0. 0. 0. 0. 0.]
     [0. 1. 0. 0. 0. 0.]
     [0. 0. 1. 0. 0. 0.]
     [0. 0. 0. 1. 0. 0.]
     [0. 0. 0. 0. 1. 0.]
     [0. 0. 0. 0. 0. 1.]]

D = [[0. 0.]
     [0. 0.]
     [0. 0.]
     [0. 0.]
     [0. 0.]
     [0. 0.]]
```

Linear quadratic regulator (LQR) design

Now that we have a linearized model of the system, we can compute another controller using linear quadratic regulator theory. As before, we wish to minimize the following cost function

$$J(\phi(\cdot), \nu(\cdot)) = \int_0^\infty \phi^T(\tau) Q \phi(\tau) + \nu^T(\tau) R \nu(\tau)\, d\tau,$$

where we have changed to our linearized coordinates:

$$\phi = z - z_e, \quad \nu = u - u_e$$

Using the standard approach for finding K, we obtain a feedback controller for the system:

$$\nu = -K\phi$$

In [13]:
```
# Start with a diagonal weighting
Q1 = np.diag([1, 1, 1, 1, 1, 1])
R1 = np.diag([1, 1])
K, X, E = ct.lqr(linsys, Q1, R1)
```

To create a controller for the system, we have to apply a control signal u, so we change back from the relative coordinates to the absolute coordinates:

$$u = -K(z - z_e) + u_e$$

Notice that, since $(Kz_e + u_e)$ is completely determined by (user-defined) inputs to the system, this term is a type of feedforward control signal.

To create a controller for the system, we can use the function `create_statefbk_iosystem()`, which creates an I/O system that takes in a desired trajectory $(x_\mathrm{d}, u_\mathrm{d})$ and the current state x and generates a control law of the form:

$$u = u_\mathrm{d} - K(x - x_\mathrm{d})$$

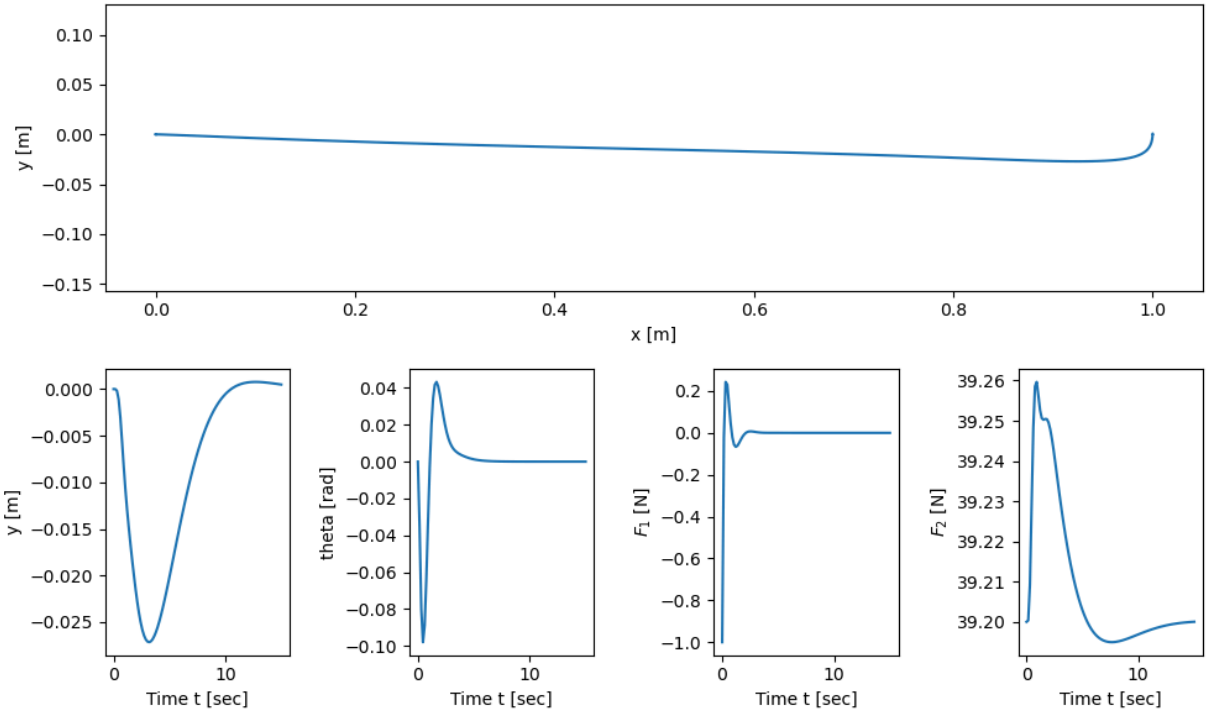
In [14]:
```
control, pvtol_closed = ct.create_statefbk_iosystem(pvtol, K)
print(control, "\n")
print(pvtol_closed)
```

```
<StateSpace>: sys[7]
Inputs (14): ['xd[0]', 'xd[1]', 'xd[2]', 'xd[3]', 'xd[4]', 'xd[5]', 'ud[0]',
'ud[1]', 'x0', 'x1', 'x2', 'x3', 'x4', 'x5']
Outputs (2): ['F1', 'F2']
States (0): []

A = []

B = []

C = []

D = [[-1.00000000e+00 -4.18744302e-07  7.85405998e+00 -1.60495816e+00
      -5.38237356e-07  2.06849834e+00  1.00000000e+00  0.00000000e+00
       1.00000000e+00  4.18744302e-07 -7.85405998e+00  1.60495816e+00
       5.38237356e-07 -2.06849834e+00]
     [-4.18744302e-07  1.00000000e+00 -2.48287512e-07 -1.36974381e-06
       2.95041664e+00  3.94965564e-08  0.00000000e+00  1.00000000e+00
       4.18744302e-07 -1.00000000e+00  2.48287512e-07  1.36974381e-06
      -2.95041664e+00 -3.94965564e-08]]


<InterconnectedSystem>: pvtol_sys[7]
Inputs (8): ['xd[0]', 'xd[1]', 'xd[2]', 'xd[3]', 'xd[4]', 'xd[5]', 'ud[0]',
'ud[1]']
Outputs (8): ['x0', 'x1', 'x2', 'x3', 'x4', 'x5', 'F1', 'F2']
States (6): ['pvtol_x0', 'pvtol_x1', 'pvtol_x2', 'pvtol_x3', 'pvtol_x4', 'pv
tol_x5']

Update: <function InterconnectedSystem.__init__.<locals>.updfcn at 0x11e7c9e
e0>
Output: <function InterconnectedSystem.__init__.<locals>.outfcn at 0x11e7c9f
80>
/Users/murray/miniconda3/envs/cds110/lib/python3.12/site-packages/control/st
atefbk.py:783: UserWarning: cannot verify system output is system state
  warnings.warn("cannot verify system output is system state")
```

Closed loop system simulation

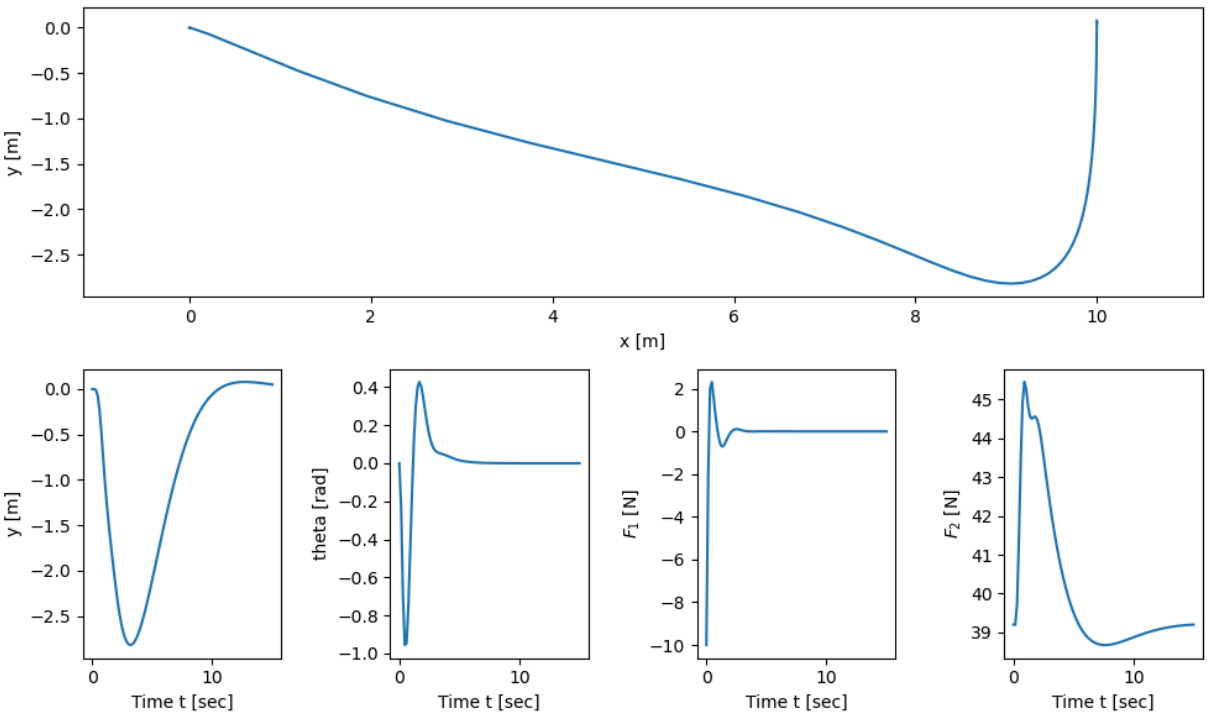
For this simple example, we set the target for the system to be a "step" input that moves the system 1 meter to the right.

In [15]:
```python
# Generate a step response by setting xd, ud
Tf = 15
T = np.linspace(0, Tf, 100)
xd = np.outer(np.array([1, 0, 0, 0, 0, 0]), np.ones_like(T))
ud = np.outer(ueq, np.ones_like(T))
ref = np.vstack([xd, ud])

response = ct.input_output_response(pvtol_closed, T, ref, xeq)
plot_results(response.time, response.states, response.outputs[6:])
```



The limitations of the linear controller can be seen if we take a larger step, say 10 meters.

In [16]:
```
xd = np.outer(np.array([10, 0, 0, 0, 0, 0]), np.ones_like(T))
ref = np.vstack([xd, ud])
response = ct.input_output_response(pvtol_closed, T, ref, xeq)
plot_results(response.time, response.states, response.outputs[6:])
```



(If time: Change values of R)