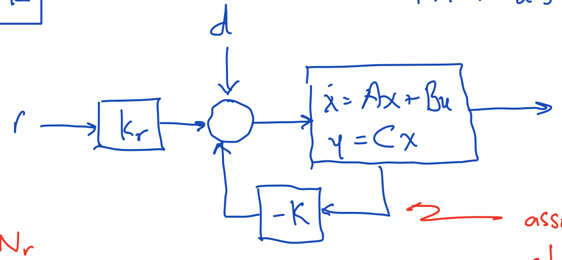
CDS 110, Lecture 4.2: State space feedback

RMM 23 Apr 2024

I. Problem setup (linear case for now)

$\dot{x} = Ax + Bu \qquad u = -Kx + k_r r$

$y = Cx \qquad \text{Goal: } y(t) \to r(t)$



assume noiseless state measurement for now

II. Eigenvalue placement

A. Reachability test: $\operatorname{rank} \overbrace{[B \;\; AB \;\cdots\; A^{n-1}B]}^{W_r} = n$

(will explore further in week 8)

B. Thm: Reachable $\Rightarrow$ can set eigenvalues arbitrarily

Pf: Special case: reachable canonical form

$$\tilde{A} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \tilde{A} - \tilde{B}\tilde{K} = \begin{bmatrix} -a_1 - k_1 & \cdots & -a_n - k_n \\ 1 & \ddots & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots a_{n-1}s + a_n \qquad \det(sI - A_{cl}) = s^n + (a_1 + k_1)s^{n-1} + \cdots$$

Claim $[B \;\; AB \cdots A^{n-1}B]$ full rank $\Rightarrow$ can find coords to put in reachable form

Pf: $z = Tx \quad T = \tilde{W}_r W_r^{-1}$ (see textbook for complete derivation)

III Reference tracking: suppose r is constant (but nonzero)

$$\dot{x} = (A - BK)x + Bk_r r \quad \Rightarrow \quad x_e = (A - BK)^{-1} B k_r r$$

$$y = Cx \qquad y_e = C(A - BK)^{-1} B k_r r \Rightarrow k_r = \frac{1}{C(A - BK)^{-1}B}$$

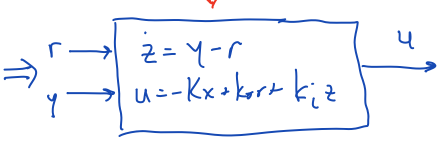
If y varies slowly compared to closed loop dynamics, this works well for $r(t)$

III Integral action: suppose d is not zero

$y_e = r + C(A - BK)^{-1} B d$ ← constant error

Fix: $u = -Kx + k_r r + k_i \int \underbrace{(y - r)}_{z} d\tau \qquad \dot{z} = y - r \quad \Rightarrow$

dynamic compensator (HW #2)



To design k_i

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\bar{B}} u \qquad u = -\bar{K} \begin{bmatrix} x \\ z \end{bmatrix} + k_r r$$

$$= -Kx + k_r r + k_i z$$

$\bar{K} = \operatorname{place}(\bar{A}, \bar{B}, \text{eigs})$ or $\bar{K} = \operatorname{lqr}(\bar{A}, \bar{B}, \bar{Q}, \bar{R})$

For r constant, can set $k_r = 0$ and rely on k_i (!)

IV Nonlinear tracking near an equil pt (x_e, u_e):

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \;\rightarrow\; \begin{aligned} z &= x - x_e \\ v &= u - u_e \end{aligned} \;\rightarrow\; \underbrace{v = -Kz + k_r(r - y_e)}_{\text{design via linearization}} \rightarrow u = u_e - K\underbrace{(x - x_e)}_{\text{deviation from } x_e} + k_r\{r - y_e\}$$